Instantons in supersymmetric gauge theories

Tobias Hansen
Talk at tuesday's "Werkstatt Seminar"

January 10, 2012

References

[1] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The Calculus of many instantons," Phys. Rept. **371** (2002) 231 [hep-th/0206063].

[2] S. Kovacs, "N=4 supersymmetric Yang-Mills theory and the AdS / SCFT correspondence (Chapter 4)," hep-th/9908171.

[3] M. A. Shifman and A. I. Vainshtein, "Instantons versus supersymmetry: Fifteen years later," In *Shifman, M.A.: ITEP lectures on particle physics and field theory, vol. 2* 485-647 [hep-th/9902018].

[4] J. Terning, "Modern supersymmetry: Dynamics and duality," (International series of monographs on physics. 132)

[5] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton, USA: Univ. Pr. (1992) 259 p

Contents

1 Introduction

The goal of this talk is to explain how to calculate VEVs of gauge theory operators in the instanton background. Doing so, one can learn something about possible nonperturbative aspects of a theory.

Remember: An instantons is a topologically nontrivial stationary point of an Euclidean finite action.

The VEV of an operator in pure a Yang–Mills theory can be calculated using the path integral:

$$\langle \mathcal{O}(x_1,\ldots,x_n)\rangle = \int [\mathcal{D}\phi] e^{-\frac{S[\phi]}{\hbar}} \mathcal{O}(\phi(x_1),\ldots,\phi(x_n)) , \tag{1}$$

The classical vacuum corresponds to a constant expectation value $\langle\phi\rangle$ for the elementary fields. A perturbative series is gained by expanding the action for small fluctuations $\delta\phi$ around the classical vacuum.

In instanton calculations one expands the action around the instanton instead of the vacuum. Let $\overline{\phi}$ be an instanton solution and

$$\phi(x) = \overline{\phi}(x) + \delta\phi(x) = \overline{\phi}(x) + \hbar^{1/2}\eta(x)$$

Expand $S[\phi]$ in powers of $\hbar$:

$$S[\phi] = S[\overline{\phi}] + \frac{\hbar}{2}\int d^4x d^4y \left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right)\bigg|_{\phi=\overline{\phi}} \eta(x)\eta(y) + O(\hbar^{3/2}) ,$$

The VEV of an operator $\mathcal{O}[\phi]$ can be evaluated in the semiclassical limit ($\hbar\to 0$) by a saddle point approximation:

$$\begin{aligned}\langle \mathcal{O}(x_1,\ldots,x_n)\rangle &= \int [\mathcal{D}\eta] e^{-\frac{S[\overline{\phi}+\eta]}{\hbar}} \mathcal{O}[(\overline{\phi}+\eta)(x_i)] = \\ &= \mathcal{O}(\overline{\phi})\, e^{-\frac{\overline{S}}{\hbar}} \left[\det\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right)\bigg|_{\phi=\overline{\phi}}\right]^{-\frac{1}{2}} (1+O(\hbar)) .\end{aligned} \tag{2}$$

In non-supersymmetric non-Abelian gauge theories the quantum corrections turn out to be divergent in the instanton background.

With supersymmetry, there are no quantum corrections to the classical vacuum, like in loop calculations a cancellation between fermions and bosons takes place. But there is a caveat: the presence of a background instanton field in general breaks supersymmetry. There are, however, so-called magic backgrounds that preserve one half of supersymmetry. This is enough to ensure the cancellation of quantum corrections.

In the gauge theories any (anti-)self-dual gluon field preserves one half of supersymmetry. But these are exactly the field configurations corresponding to (anti-)instantons!

I will start with a simpler (non-gauge) example where the conservation of one half of supersymmetry and subsequent cancellation of quantum corrections also takes place: the domain wall in the minimal Wess–Zumino model. Then I will discuss non-supersymmetric Yang–Mills instantons and sketch how SUSY instantons are constructed out of them. The section on instanton calculus outlines how the path integral calculation is performed. If there is enough time left, I will discuss in the end how the holomorphic gauge couling obtains instanton corrections. This is somewhat out of the line of this talk, but it is a good preparation for the following talks, where holomorphy arguments will play an important role.

1.1 Preliminary: Superspace and superfields

The four-dimensional space x^μ can be promoted to superspace by adding four Grassmann coordinates θ_α and $\bar\theta_{\dot\alpha}$, (α, $\dot\alpha = 1,2$). The coordinate transformations

$$(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \to (x^\mu + i\theta\sigma^\mu\bar\varepsilon - i\varepsilon\sigma^\mu\bar\theta, \theta_\alpha + \varepsilon_\alpha, \bar\theta_{\dot\alpha} + \bar\varepsilon_{\dot\alpha}) \tag{3}$$

(with $\sigma^\mu = \{1, \vec\tau\}$, $\vec\tau$ Pauli the matrices) add SUSY to the translational and Lorentz transformations.

Since the SUSY transformation mixes between bosonic and fermionic fields, single bosonic or fermionic fields can not be SUSY invariant. SUSY invariant fields are called Superfields and contain always both bosonic and fermionic components.

The minimal supermultiplet of fields includes one complex scalar field $\phi(x)$ (two bosonic states) and one complex Weyl spinor $\psi^\alpha(x)\,,\ \alpha = 1,2$ (two fermionic states). Both fields are united in one *chiral superfield*,

$$\Phi(x_L, \theta) = \phi(x_L) + \sqrt{2}\theta^\alpha\psi_\alpha(x_L) + \theta^2 F(x_L)\,, \tag{4}$$

It is called chiral because it is defined on the first of the two invariant subspaces $\{x^\mu_L, \theta_\alpha\}$ and $\{x^\mu_R, \bar\theta_{\dot\alpha}\}$ of superspace ($x^\mu_{L,R} = x^\mu \mp i\theta\sigma^\mu\bar\theta$). It's conjugate $\bar\Phi(x_R, \bar\theta)$ is defined on the other subspace and is thus called antichiral. F is an auxiliary component, that means it appears in the Lagrangian without the kinetic term.

2 Instanton solutions

2.1 Supersymmetric domain wall

2.1.1 Preliminary: The minimal Wess-Zumino model

The minimal Wess-Zumino model contains one chiral superfield $\Phi(x_L, \theta)$ and its complex conjugate $\bar\Phi(x_R, \bar\theta)$. The action of the model is

$$S = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \Phi\bar\Phi + \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{W}(\Phi) + \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^2\bar\theta\, \bar{\mathcal{W}}(\bar\Phi)\,. \tag{5}$$

Note that the first term is the integral over the full superspace, while the second and the third run over the chiral subspaces. The holomorphic function $\mathcal{W}(\Phi)$ is called the *superpotential.* In components the Lagrangian has the form

$$\mathcal{L} = (\partial^\mu\bar\phi)(\partial_\mu\phi) + \psi^\alpha i\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} + \bar F F + \left\{ F\,\mathcal{W}'(\phi) - \frac{1}{2}\mathcal{W}''(\phi)\psi^2 + \text{H.c.}\right\}\,. \tag{6}$$

From Eq. (6) it is obvious that F can be eliminated by virtue of the classical equation of motion,

$$\bar{F} = -\frac{\partial\,\mathcal{W}(\phi)}{\partial\phi}\,, \tag{7}$$

so that the *scalar potential* describing self-interaction of the field ϕ is

$$V(\phi,\bar{\phi}) = \left|\frac{\partial\,\mathcal{W}(\phi)}{\partial\phi}\right|^2 . \tag{8}$$

If one limits oneself to renormalizable theories, the superpotential $\mathcal{W}$ must be a polynomial function of Φ of power not higher than three. In the model at hand, with one chiral superfield, the generic superpotential can be always reduced to the following "standard" form

$$\mathcal{W}(\Phi) = \frac{m^2}{\lambda}\,\Phi - \frac{\lambda}{3}\Phi^3\,. \tag{9}$$

The quadratic term can be always eliminated by a redefinition of the field Φ. Moreover, by using the R symmetries one can always choose the phases of the constants m and λ at will.

Let us study the set of classical vacua of the theory, *the vacuum manifold*. In the simplest case of the vanishing superpotential, $\mathcal{W}=0$, any coordinate-independent (coordinates in superspace) field $\Phi_{\rm vac} = \phi_0$ can serve as a vacuum. The vacuum manifold is then the one-dimensional (complex) manifold $C^1 = \{\phi_0\}$. The continuous degeneracy is due to the absence of the potential energy, while the kinetic energy vanishes for any constant ϕ_0.

This continuous degeneracy is lifted by the superpotential. In particular, the superpotential (9) implies two classical vacua,

$$\phi_{\rm vac} = \pm\frac{m}{\lambda}\,. \tag{10}$$

Thus, the continuous manifold of vacua C^1 reduces to two points.

2.1.2 Supersymmetric domain wall

Field configurations interpolating between two degenerate vacua are called the *domain walls*. They have the following properties: (i) the corresponding solutions are static and depend only on one spatial coordinate; (ii) they are topologically stable and indestructible – once a wall is created it cannot disappear. Assume for definiteness that the wall lies in the xy plane. Then the wall solution $\phi_{\rm w}$ will depend only on z. Since the wall extends indefinitely in the xy plane, its energy $E_{\rm w}$ is infinite.

The classical equation of motion is

$$0 = \partial_z^2\phi - 2\bar{\lambda}\bar{\phi}\Big(\frac{m^2}{\lambda} - \lambda\phi^2\Big)\,. \tag{11}$$

The solution is

$$\phi_{\rm w} = \frac{m}{\lambda}\tanh(|m|z)\,. \tag{12}$$

Note that the parameters m and λ are not assumed to be real. A remarkable feature of this solution is that it preserves one half of supersymmetry. Indeed, the SUSY transformations (3) generate the following transformation of fields,

$$\delta\phi = \sqrt{2}\varepsilon\psi\,, \qquad \delta\psi^\alpha = \sqrt{2}\left[\varepsilon^\alpha F + i\,\partial_\mu\phi\,(\sigma^\mu)^{\alpha\dot\alpha}\,\bar\varepsilon_{\dot\alpha}\right]\,. \tag{13}$$

The domain wall we consider is purely bosonic, $\psi = 0$. Moreover,

$$\partial_z\phi_{\rm w}(z) = |m|\frac{m}{\lambda}(1-\tanh^2(|m|z)) = e^{i\eta}\left.\frac{\partial\bar{\mathcal W}}{\partial\bar\phi}\right|_{\bar\phi=\phi^*_{\rm w}} = -e^{i\eta}\,F \tag{14}$$

where

$$\eta = \arg\frac{m^3}{\lambda^2}\,. \tag{15}$$

The relation (14) means that the domain wall actually satisfies the first order differential equation, which is by far a stronger constraint than the classical equations of motion. Due to this feature

$$\delta\psi_\alpha \propto \varepsilon_\alpha + i\,e^{i\eta}\,(\sigma^z)_{\alpha\dot\alpha}\,\bar\varepsilon^{\dot\alpha} \tag{16}$$

vanishes provided that

$$\varepsilon_\alpha = -i\,e^{i\eta}\,(\sigma^z)_{\alpha\dot\alpha}\,\bar\varepsilon^{\dot\alpha}\,. \tag{17}$$

This condition singles out two supertransformations (out of four) which do not act on the domain wall.

Now, let us calculate the wall tension at the classical level. To this end we rewrite the expression for the tension as

$$\begin{aligned}
\mathcal{E} &= \int_{-\infty}^{+\infty} {\rm d}z\,\left[\partial_z\bar\phi\,\partial_z\phi + \bar F F\right] && (18)\\
&= \int_{-\infty}^{+\infty} {\rm d}z\,\left\{\left|\,\partial_z\phi + e^{i\eta}\,F\,\right|^2 + \left[e^{-i\eta}\,\frac{\partial\phi}{\partial z}\frac{\partial\mathcal W}{\partial\phi} + {\rm H.c.}\right]\right\} && (19)\\
&= \int_{-\infty}^{+\infty} {\rm d}z\,\left\{\left|\,\partial_z\phi + e^{i\eta}\,F\,\right|^2 + \left[e^{-i\eta}\,\partial_z\mathcal W + {\rm H.c.}\right]\right\}\,, && (20)
\end{aligned}$$

where $F = -\partial\bar{\mathcal W}/\partial\bar\phi$ and it is implied that ϕ depends only on z.

For the wall tension $\mathcal{E}_{\rm w}$ the first term vanishes and all we need to calculate is $\mathcal W$ at $\phi_{\rm w}(z=\pm\infty) = \pm\frac{m}{\lambda}$. The result is

$$\mathcal{E}_{\rm w} = e^{-i\eta}\left\{\mathcal W(\phi_{\rm w}(z=\infty)) - \mathcal W(\phi_{\rm w}(z=-\infty))\right\} + {\rm H.c.} = \frac{8}{3}\left|\frac{m^3}{\lambda^2}\right|\,. \tag{21}$$

The wall tension coincides with the modulus of the topological charge $\mathcal Z$

$$\mathcal{E}_{\rm w} = |\mathcal Z|\,, \tag{22}$$

which is defined as

$$\mathcal Z = 2\left\{\mathcal W(\phi(z=\infty)) - \mathcal W(\phi(z=-\infty))\right\} = \frac{8\,m^3}{3\,\lambda^2}\,. \tag{23}$$

Note that the phase of $\mathcal Z$ coincides with η introduced in Eq. (15).

Such states which satisfy the first order equations and have masses that coincide with the topological charges are called BPS or *BPS-saturated*. BPS stands for Bogomolny, Prasad and Sommerfield.

2.2 Instantons in Yang–Mills theories without SUSY

Before discussing instantons in supersymmetric theories, the pure Yang–Mills instanton has to be adressed, because it is needed for the construction of the SUSY instanton.

The Euclidean Yang–Mills action is

$$S_{\text{YM}} = \int d^4x \, \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{24}$$

Since solutions of the classical equations of motion minimize the action, the can be found by calculating the lower bound of the action.

$$0 \leq \int d^4x \, \text{Tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 = \int d^4x \, \text{Tr}(2F^2 \pm 2F\tilde{F}) \tag{25}$$

$$\Rightarrow \int d^4x \, \text{Tr}F^2 \geq \left| \int d^4x \, \text{Tr}F\tilde{F} \right| = 16\pi^2|n| \, , \tag{26}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength and n is the winding number of the gauge field. So the Euclidean action is minimized for

$$F^{\mu\nu} = \pm\tilde{F}^{\mu\nu} \, . \tag{27}$$

The solution of this equation for the SU(2) gauge theory and winding number 1 is the BPST-instanton

$$A_n = g^{-1}\frac{2(x-X)_m\sigma_{mn}}{(x-X)^2+\rho^2} \, , \tag{28}$$

with $\sigma_{mn} = \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m)$. ρ and X_m are collective coordinates that parametrize different instanton solutions. ρ is the instanton size and X_m its position. The SU(2) BPST 1-instanton has in total 8 collective coordinates, a general n-Instanton in the SU(N) theory has 4nN. One can find all instantons of a pure Yang–Mills theory by the ADHM construction (Atiyah, Drinfel'd, Hitchin, Manin 1978).

2.3 Instantons in supersymmetric gauge theories

2.3.1 Preliminary: SUSY gauge theory

To get a supersymmetric gauge theory, we have to introduce a SUSY analogue to a vector field. This is the vector superfield V, which is defined as a real superfield

$$V = V^\dagger \, . \tag{29}$$

A general vector superfield has many components, one of them being a vector field

$$V = \cdots + \theta\sigma^\mu\bar{\theta}A_\mu + \ldots \, . \tag{30}$$

One can also make a vector superfield out of a chiral superfield Φ by taking $\Phi + \Phi^\dagger$. This contains a term

$$\Phi + \Phi^\dagger = \cdots + \theta\sigma^\mu\bar{\theta}\,(i\partial\mu(\phi - \phi^*)) + \ldots \, . \tag{31}$$

Now $V \to V + \Phi + \Phi^\dagger$ takes $A_\mu \to A_\mu + \partial\mu\Lambda$. This is the SUSY version of an Abelian gauge transformation. It can be used to put V into Wess-Zumino gauge, where for $\mathcal{N} = 1$ only the following component fields are left:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2(\bar{\theta}\bar{\lambda}) - i\bar{\theta}^2(\theta\lambda) + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{32}$$

Here A_μ is the gauge boson λ and $\bar{\lambda}$ are gauginos and D is an auxiliary field. Now we need a gauge invariant and supersymmetric field strength so that we can add kinetic terms to the lagrangian. One can construct the following chiral superfield out of the components of V:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\theta_\beta F_{\mu\nu} + \theta^2\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} \tag{33}$$

Squaring this, we almost arrive at the superpotential for the SUSY Yang–Mills action:

$$W^\alpha W_\alpha|_{\theta^2} = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}F^{\mu\nu}\tilde{F}_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 \tag{34}$$

We also introduce the *holomorphic gauge coupling*

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2} \tag{35}$$

The SUSY Yang–Mills action is

$$\frac{1}{16\pi i}\int d^4x d^2\theta\, \tau\, W^\alpha W_\alpha + \text{H.c.} \tag{36}$$

$$= \int d^4x \left[-\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} - \frac{\theta_{\text{YM}}}{32\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} + \frac{i}{g^2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{2\pi^2}{g^2}D^2\right]. \tag{37}$$

2.3.2 Construction of SUSY instantons

For the this section a slightly more general (to cover also $\mathcal{N} = 2$ and $\mathcal{N} = 4$) and Euclidean SUSY Yang–Mills action is used. It is

$$S = \int d^4x\, \text{Tr}\left\{-\frac{1}{2}F_{mn}^2 - \frac{i\theta_{\text{YM}}g^2}{16\pi^2}F_{mn}\tilde{F}_{mn} - 2\mathcal{D}_n\bar{\lambda}_A\bar{\sigma}_n\lambda^A + \mathcal{D}_n\phi_a\mathcal{D}_n\phi_a \right. \tag{38}$$

$$\left. -g\bar{\lambda}_A\Sigma_a^{AB}[\phi_a, \bar{\lambda}_B] - g\lambda^A\Sigma_{aAB}[\phi_a, \lambda^B] - \frac{1}{2}g^2[\phi_a, \phi_b]^2\right\} \tag{39}$$

$A = 1, .., \mathcal{N}$ is an R-Symmetry index and Σ_a^{AB} are associated to the different R-Symmetry groups for different $\mathcal{N}$. $\mathcal{D}$ are covariant derivatives. ϕ_a, $a = 1, ..., 2(\mathcal{N} - 1)$ are real scalar fields that are part of the vector supermultiplets for $\mathcal{N} \in \{2, 4\}$. For the $\mathcal{N} = 1$ case it is set to zero.

The equations of motions for $\mathcal{N} = 1$ are thus

$$\mathcal{D}_m F_{nm} = 2g\bar{\sigma}_n\{\lambda, \bar{\lambda}\} \tag{40}$$

$$\bar{\not{\mathcal{D}}}\lambda = 0 \tag{41}$$

$$\not{\mathcal{D}}\bar{\lambda} = 0 \tag{42}$$

The ADHM instanton with $\lambda = \bar{\lambda} = 0$ is a solution of these equations. To get the general solution, 41 and 42 have to be evaluated in the ADHM background ($A_m = A_{m,\mathrm{ADHM-instanton}}$) of topological charge n. It can be shown that 41 has in general $2nN$ linearly independent solutions Λ_i with $i = 1, ..., 2nN$, while 42 has no nontrivial solutions for ($n > 0$). So for the general and exact $\mathcal{N} = 1$ instanton

$$\bar{\lambda} = 0 \tag{43}$$

$$\lambda = \sum_{i=1}^{2nN} \psi^i \Lambda_i \tag{44}$$

The source term in 40 vanishes because of 43, so the vector field stays the same as in pure Yang-Mills theory. The ψ^i are Grassmann valued collective coordinates.

For $N \in \{2, 4\}$ one can also consider fluctuations around the ADHM instanton, but now also the scalar fields ϕ_a have to be considered. The equations of motion are now

$$\mathcal{D}_m F_{nm} = 2g\bar{\sigma}_n\{\lambda^A, \bar{\lambda}_A\} + 2g[\phi_a, \mathcal{D}_n\phi_a] \tag{45}$$

$$\bar{\not{\mathcal{D}}}\lambda^A = g\Sigma_a^{AB}[\phi_a, \bar{\lambda}_B] \tag{46}$$

$$\not{\mathcal{D}}\bar{\lambda}_A = g\bar{\Sigma}_{aAB}[\phi_a, \lambda^B] \tag{47}$$

$$\mathcal{D}^2\phi_a = g^2[\phi_b, [\phi_b, \phi_a]] + g\bar{\Sigma}_{aAB}\lambda^A\lambda^B + g\Sigma_a^{AB}\bar{\lambda}^A\bar{\lambda}^B \tag{48}$$

One can proceed perturbatively order by order in the coupling. For $\mathcal{N} = 2$ an exact solution can be obtained for the case that the scalar VEVs are zero. For $\mathcal{N} = 4$ no exact solution is known. Approximate solutions are called quasi-instantons.

3 Instanton calculus

In the introduction it was discussed how the path integral can be calculated using the saddle point approximation for one instanton. In the previous section however a continuum of instantons was found, parameterized by the collective coordinates. As before, the fields are expanded around the instantons,

$$A_n(x) = A_n(x; X) + \delta A_n(x; X)\,. \tag{49}$$

Since the total field does not depend on the collective coordinates X, both the instanton and the fluctuations do. The saddle point approximation can only be performed over the non-zero modes, so the fluctuations are split in zero modes $\delta_\mu A_n$ and non-zero modes $\tilde{A}_n$:

$$\delta A_n(x; X) = \sum_\mu \xi^\mu \delta_\mu A_n + \tilde{A}_n\,. \tag{50}$$

Now the integrals over the zero and non-zero modes of the gauge field are separated in the path integral

$$\int [dA_n] = \int \left\{ \sqrt{\det g(X)} \prod_\mu \frac{d\xi^\mu}{\sqrt{2\pi}} \right\} [d\tilde{A}_n]\,. \tag{51}$$

$g(X)$ is the metric on the moduli space

$$g_{\mu\nu}(X) = -2 \int d^4x \, \mathrm{Tr} \delta_\mu A_n(x;X) \delta_\nu A_n(x;X) \, . \tag{52}$$

The chiral fermion is split into zero and non-zero modes (it is expanded around 0).

$$\lambda^A(x) = \lambda^{(0)A}(x; X, \psi^A) + \tilde{\lambda}^A(x; X, \psi^A) \, . \tag{53}$$

The anti-chiral fermions and scalar fields have no zero modes in the instanton background, so they keep their names and are treated like the non-zero modes. The fermionic integrals are separated

$$\int \prod_{A=1}^{\mathcal{N}} [d\lambda^A][d\bar{\lambda}_A] \propto \int \prod_{A=1}^{\mathcal{N}} \left\{ \prod_{i=1}^{2nN} d\psi^{iA} [d\tilde{\lambda}^A][d\bar{\lambda}_A] \right\} \, . \tag{54}$$

The integral over the non-zero modes can now be performed using the saddle point method like in the introduction. The action is expanded around the instanton:

$$S[A_n + A_n^{(0)} + \tilde{A}_n, \lambda^{(0)A} + \tilde{\lambda}^A, \bar{\lambda}_A, \phi_a] = -2\pi i n \tau + S_{\mathrm{kin}} + S_{\mathrm{int}} \, . \tag{55}$$

S_{kin} denotes the kinetic terms for the non-zero mode fluctuations and S_{int} includes interactions between zero and non-zero modes.

Now the integration over the non-zero modes $\{\tilde{A}_n, \tilde{\lambda}^A, \bar{\lambda}_A, \phi_a\}$ and the ghosts $\{b, c\}$, which arise from Pauli-Villars regularization, can be performed. This defines the instanton effective action S_{eff} on the collective coordinates:

$$\mathrm{e}^{-S_{\mathrm{eff}}} := \mathrm{e}^{2\pi i n \tau} \int [d\tilde{A}][db][dc][d\tilde{\lambda}][d\bar{\lambda}][d\phi] \mathrm{e}^{-S_{\mathrm{kin}} - S_{\mathrm{int}} - S_{\mathrm{gh}}} \tag{56}$$

It turns out that S_{eff} vanishes if the instanton is an exact solution. For $\mathcal{N} = 4$ and $\mathcal{N} = 2$ with non-vanishing VEVs, it can be determined perturbatively in g using the Feynman rules emerging from S_{kin} and S_{int}.

The integrals over the expansion coefficients of the zero modes ξ^μ can be traded for integrals over the collective coordinates. The derivatives of the instanton with respect to the collective coordinates $\partial A_n / \partial X^\mu$ are tangent vectors pointing towards directions in which the action stays minimized. This means they are a mix of zero modes and local gauge transformations. If all the $\partial A_n / \partial X^\mu$ are orthogonal to the gauge transformations, one can simply replace $\xi^\mu \delta_\mu A_n$ by $X^\mu \partial A_n / \partial X^\mu$. Otherwise a few more steps are needed.

4 The holomorphic gauge coupling

Now let us determine the scale dependence of the holomorphic gauge coupling τ. It will turn out that it gets no perturbative corrections, but instead non-perturbative instanton corrections.

The one-loop running is given by the renormalization group equation

$$\mu\frac{dg}{d\mu} = -\frac{b}{16\pi^2}g^3\,, \tag{57}$$

where for an SU(N) gauge theory with F flavours and $\mathcal{N}=1$ SUSY

$$b = 3N - F\,. \tag{58}$$

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\ln\left(\frac{|\Lambda|}{\mu}\right)\,. \tag{59}$$

$|\Lambda|$ is the intrinsic scale of the non-Abelian gauge theory. It is written as a modulus, because it will be turned into a complex number next.

The one-loop running version of τ is

$$\tau_{1-\text{loop}} = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2(\mu)} = \frac{\theta_{\text{YM}}}{2\pi} + \frac{b}{2\pi i}\ln\left(\frac{|\Lambda|}{\mu}\right) = \frac{b}{2\pi i}\ln\left(\frac{|\Lambda|}{\mu}\mathrm{e}^{\frac{i\theta_{\text{YM}}}{b}}\right)\,. \tag{60}$$

The holomorphic intrinsic scale is now defined as

$$|\Lambda|\mathrm{e}^{\frac{i\theta_{\text{YM}}}{b}}\,, \tag{61}$$

so

$$\tau_{1-\text{loop}} = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right)\,. \tag{62}$$

Complex coefficients in the superpotential such as couplings or here Λ can be regarded as background fields and so the superpotential also has to be a holomorphic function of Λ.

One can now determine how instanton corrections depend on Λ. The Euclidean action is minimized by setting $\lambda = \bar{\lambda} = D = 0$ and $F^{\mu\nu} = \tilde{F}^{\mu\nu}$, as in the non-SUSY case. Then the n-instanton action $S_{\text{n}-\text{inst}}$ becomes

$$S_{\text{n}-\text{inst}} = \frac{4\pi^2 n}{g^2(\mu)} - i\theta_{\text{YM}}n = -2\pi i n\tau_{1-\text{loop}}\,. \tag{63}$$

This leads to n-instanton effects being proportional to

$$\mathrm{e}^{-S_{\text{n}-\text{inst}}} = \mathrm{e}^{2\pi i n\tau_{1-\text{loop}}} = \left(\frac{\Lambda}{\mu}\right)^{bn}\,. \tag{64}$$

Let's get to τ being holomorphic in Λ. It can thus be written as

$$\tau(\Lambda;\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) + f(\Lambda;\mu)\,, \tag{65}$$

with $f(\Lambda;\mu)$ holomorphic in Λ. In the weak coupling limit $\Lambda \to 0$ this must reduce to the one-loop solution, so $f(\Lambda;\mu)$ must have a Taylor series in positive powers of Λ.

Looking at the path integral

$$\int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}D\mathrm{e}^{iS}\,, \tag{66}$$

one sees that $\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} + 2\pi$ is a symmetry of the theory, because S depends only on θ_{YM} through the term which is an integer times θ_{YM}. The symmetry transformation can also be expressed as $\Lambda \to \mathrm{e}^{\frac{2\pi i}{b}}\,\Lambda$. To respect this symmetry, the Taylor series must be in powers of Λ^b. This leaves us with the final result

$$\tau(\Lambda;\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{bn}\,, \tag{67}$$

The new terms have the form of n-instanton corrections. There are no further perturbative corrections.