Instanton-Driven Saturation at Small x

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in collaboration with

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[F. Sch. & A.Utermann, Phys. Lett. **B543** (2002) 197;
Acta Phys. Polon. **B33** (2002) 3633;
Proc. SEWM 2002, hep-ph/0301177.]

– Ringberg Castle, Oct 1st, 2003 –

1. Motivation

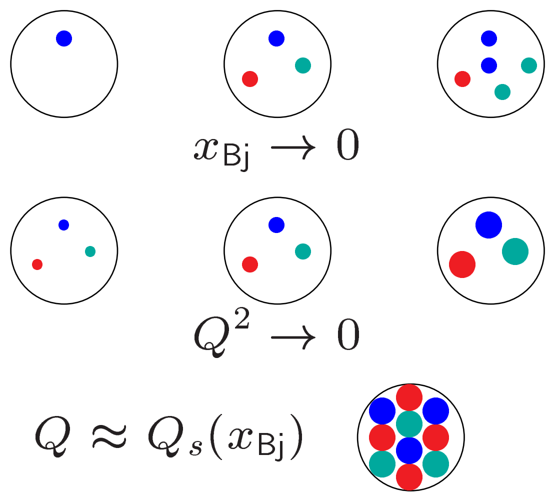
Saturation in the parton picture

- In eP-scattering at HERA: **Strong rise** of gluon distribution towards small x_{Bj}
  - Rise due to DGLAP equation (confirmation of QCD)
  - However: Undamped rise → violation of unitarity ($\sigma \leq \mathcal{O}\left(\log^2 \frac{1}{x_{\mathrm{Bj}}}\right)$ [Froissart '61])
  - Reason: Linear DGLAP or BFKL equations neglect interactions between partons

- Number of partons rises with decreasing x_{Bj}

  "size" $\sim \frac{1}{Q}$ of partons rises with decreasing Q^2

  Partons in the proton start to **overlap**



- **Interaction** between partons
  → non-linear corrections to the evolution equations
  [Gribov, Levin & Ryskin '81-'83, Mueller & Qiu '86]

- **Idea**: Interaction → rise of the gluon distribution at small x_{Bj} is tamed → gluon distribution **"saturates"**.

Instantons and Saturation?

- Small-$x_{\rm Bj}$ rise of parton densities $\rightarrow$ **novel regime of QCD**: α_s small but parton density large $\rightarrow$ breakdown of pQCD!

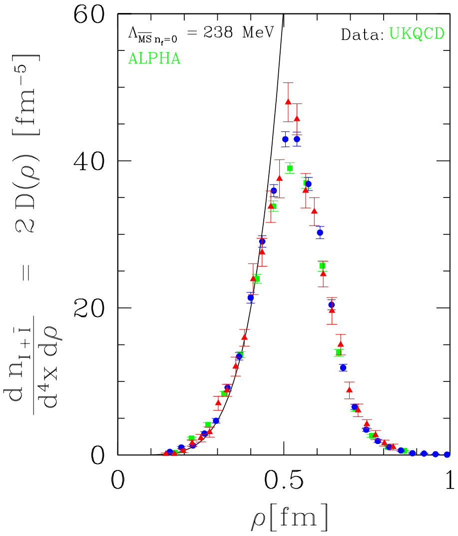
- [**McLerran** *et al* '01]: Saturation state, multi-gluon state of high occupation number $\rightarrow$ "**colour-glass condensate**", associated with a strong classical, colour field $A_\mu \propto \frac{1}{\sqrt{\alpha_s}}$

- In this regime, **instantons (I)** very interesting:
  - Non-perturbative colour fields appropriate!
  - I's $\Leftrightarrow$ known, strong classical gluon fields ($A_\mu^{(I)} \propto \frac{1}{\sqrt{\alpha_s}}$)
  - "**colour-glass condensate**" $\Leftrightarrow$ QCD -sphaleron state??

- Further **arguments** for instanton-driven saturation:

– **I-perturbation theory in DIS**:
I-contribution grows strongly towards smaller $x_{\rm Bj}$ and Q^2
[A. Ringwald & F. Sch. '94, '98]

– **Lattice information**:
intrinsic instanton scale: $\langle\rho\rangle \approx 0.5$ fm



- **Central questions**:
  - Are there saturation phenomena in an I- background ?
  - Is the saturation scale Q_s connected with $\langle\rho\rangle$?

Plan

2. **Setting the Stage**

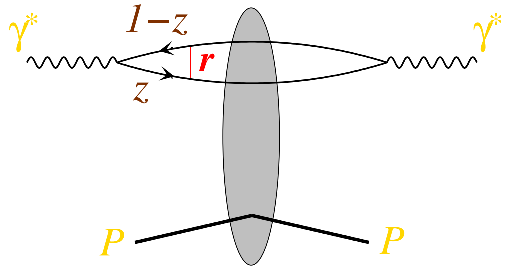
3. **From I-perturbation Theory to Saturation**

4. **Wilson-loop Scattering in an I-Background**

5. **Conclusions and Outlook**

2. Setting the Stage

- Investigation of saturation in I-background most transparent in the colour-dipole picture:



z: longitudinal photon momentum fraction carried by the quark

$\boldsymbol{r}$: transverse $(q\bar{q})$-size

[Nikolaev & Zakharov '90; Mueller '94]

- **Intuitively in the proton (P)-rest frame:** for small $x_{\rm Bj}$, γ^* fluctuates mainly into $q\bar{q}$-dipole far upstream with

$$\tau_{q\bar{q}-\text{formation}} \gg \tau_{(q\bar{q})\,P-\text{interaction}} \rightarrow \text{factorisation}$$

$$\boxed{\sigma_{L,T}(x_{\rm Bj}, Q^2) = \int_0^1 dz \int d^2\boldsymbol{r}\ |\Psi_{L,T}^{\gamma^*\rightarrow q\bar{q}}(z, \boldsymbol{r})|^2\, \sigma_{\rm DP}(\boldsymbol{r}, \ldots)}$$

- Familiar photon wave function, $\Psi_{L,T}^{\gamma^*\rightarrow q\bar{q}}$: calculable in pQCD

- No direct proof of saturation. But successful models **[GBW]** incorporating saturation that describe the data efficiently:

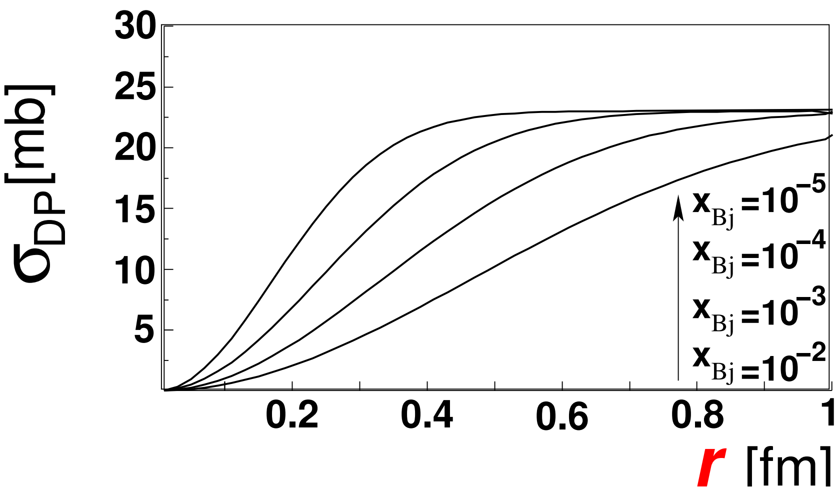
– **Colour transparency**:

$$\sigma_{\rm DP} \overset{r\rightarrow 0}{\propto} \pi\, r^2$$

[Low '75; Nikolaev & Zakharov '90]

– **Saturation**:

$$\sigma_{\rm DP} \overset{r \gtrsim 0.5\text{fm}}{\longrightarrow} \textbf{constant}$$



Strategy

- **Guiding Question:** Can background instantons of size $\sim \langle\rho\rangle$ give rise to a "saturating", geometrical form ?

$$\sigma^{(I)}_{\rm DP}(r,\ldots) \overset{r \gtrsim \langle\rho\rangle}{\sim} \pi\langle\rho\rangle^2$$

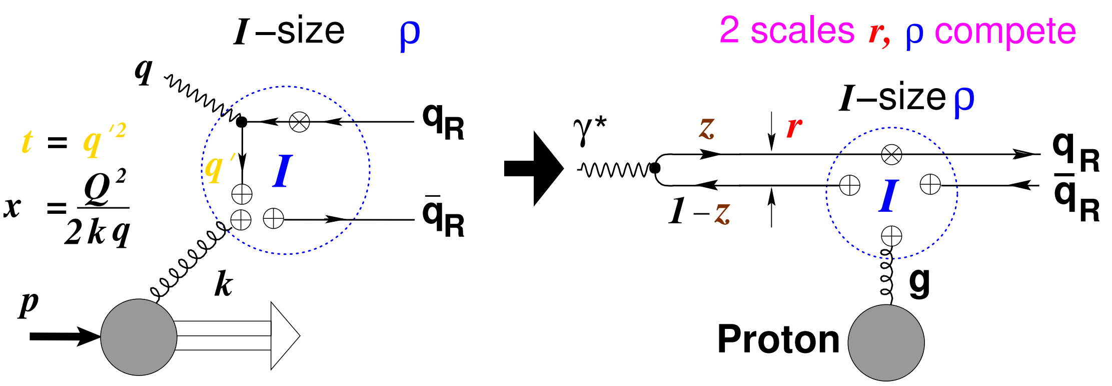
Answers from two **different approaches**:

1. **From I-perturbation theory to saturation**:
   - Start: Large Q^2 & cuts $\rightarrow$ I-pert. theory strictly valid
   - Transform known results on I-induced **DIS** processes [Moch, Ringwald & F. Sch. '97; Ringwald & F. Sch. '98] into colour-dipole picture
   - Increase $q\bar{q}$-dipole size r carefully towards hadronic dimensions with help of lattice results
   - $\xrightarrow{\text{lattice}}$ Cuts are now longer necessary!

2. **Wilson-loop scattering in an I-background**
   - Semi-classical approach: Non-abelian **eikonal approximation**: $q\bar{q}$-dipole $\Leftrightarrow$ Wilson loop
   - Proton $\Leftrightarrow$ source of (non-perturbative) colour field, instantons as a concrete example, analytically calculable!

3. From I-perturbation Theory to Saturation

Simplest Process $\gamma^* + g \overset{(I)}{\to} q_{\rm R} + \overline{q}_{\rm R}$



- Starting point: [Moch, Ringwald & Schrempp, Nucl. Phys. **B507** (1997) 134]

$$\sigma_{L,T}^{\gamma^* P}(x_{\rm Bj}, Q^2) = \int_{x_{\rm Bj}}^1 \frac{dx}{x}\left(\frac{x_{\rm Bj}}{x}\right) G\left(\frac{x_{\rm Bj}}{x}, \mu^2\right) \int dt\, \frac{d\hat{\sigma}_{L,T}^{\gamma^* g}}{dt}(x, t, Q^2)$$

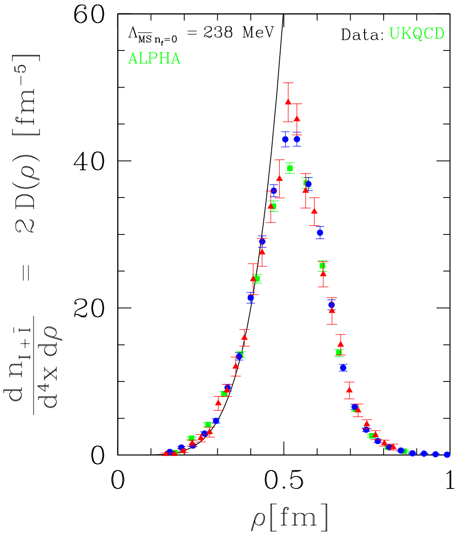
e.g.: $\frac{d\hat{\sigma}_L^{\gamma^* g}}{dt} \overset{(I)}{=} \frac{\pi^7}{2}\frac{e_q^2}{Q^2}\frac{\alpha_{\rm em}}{\alpha_s}\left[x(1-x)\sqrt{tu}\,\frac{\mathcal{R}(\sqrt{-t})-\mathcal{R}(\mathcal{Q})}{Q^2+t} - (t \leftrightarrow u)\right]^2$
with the integral, $\mathcal{R}(\mathcal{Q}) = \int_0^\infty d\rho\, D(\rho)\rho^5\, (\mathcal{Q}\rho){\rm K}_1(\mathcal{Q}\rho)$

- In usual I-perturbation theory, ***I*-size distribution**
  $D(\rho) = D_{I\text{-pert.}}(\rho) \propto \rho^{6-\frac{2}{3}n_f}$, only valid for $\rho \lesssim 0.35$ fm
- For sufficiently large $\mathcal{Q}$, $(\mathcal{Q}\rho){\rm K}_1(\mathcal{Q}\rho) \sim e^{-\mathcal{Q}\rho}$ suppresses large instantons $\to$ I-pert. theory applicable
- $\mathcal{R}(\mathcal{Q})$ is starting point for continuation to smaller $\mathcal{Q}$

– **Crucial Lattice information**:

D_{lattice} strongly peaked around average I-size: $\langle\rho\rangle \approx 0.5$ fm

– **Strategy:** Replace $D_{I\text{-pert.}}$ by D_{lattice}



- $\Rightarrow$**Note:** $\boxed{\mathcal{R}(0) = \int_0^\infty d\rho\, D_{\text{lattice}}(\rho)\,\rho^5 \approx 0.3\ \text{fm} \quad \textbf{finite}}$

- Starting point of calculation: $k_\perp$-factorisation

  – In this talk: leading contributions (order $k_\perp^2\, r^2$)
  $\rightarrow$ Integration of unintegrated gluon distribution:

$$x\, G(x, \mu^2) = \int_0^{\mu^2} \frac{d\boldsymbol{k}^2}{\boldsymbol{k}^2}\, \mathcal{F}(x, \boldsymbol{k}^2)$$

  – Similar to calculation within pQCD (two-gluon exchange)

- Using variable transformation and 2d-Fourier transformation $\boxed{(x, t)} \Leftrightarrow \boxed{(z, \boldsymbol{r})}$,

one indeed obtains the colour-dipole structure:

$$\boxed{\int_{x_{\text{Bj}}}^1 \frac{d\,x}{x}\{\ldots\} \int dt\, \frac{d\hat{\sigma}_{L,T}^{\gamma^* g}}{dt} \Rightarrow \int dz \int d^2\boldsymbol{r}\ \left(|\Psi_{L,T}|^2\, \sigma_{\text{DP}}^{(I)}\right)}$$

- **Interesting:** Background instanton induces additional length scale, I-size $\rho \approx \langle\rho\rangle$ (besides $q\bar{q}$-dipole size r)
- As expected intuitively, the 2 scales $r,\ \rho$ "compete": (with $\hat{Q} = \sqrt{z(1-z)}\,Q$)

e.g.
$$\left(|\Psi_L|^2\,\sigma_{\rm DP}\right)^{(I)} \approx |\,\Psi_L^{\rm pQCD}(z,r)\,|^2\,\frac{1}{\alpha_{\rm s}}\,x_{\rm Bj}\,G(x_{\rm Bj},\mu^2)\,\frac{\pi^8}{12}$$
$$\times\left\{\int_0^\infty d\rho\,D_{\rm lattice}(\rho)\,\rho^5\left(\frac{-\frac{d}{dr^2}\left(2r^2\frac{{\rm K}_1(\hat{Q}\sqrt{r^2+\rho^2/z})}{\hat{Q}\sqrt{r^2+\rho^2/z}}\right)}{{\rm K}_0(\hat{Q}r)}-(z\leftrightarrow 1-z)\right)\right\}^2$$

- Note: $-\frac{d}{dr^2}\left(2r^2\frac{K_1\left(\hat{Q}\sqrt{r^2+\rho^2/z}\right)}{\hat{Q}\sqrt{r^2+\rho^2/z}}\right) \approx \begin{cases} -\frac{K_1\left(Q\rho\sqrt{1-z}\right)}{Q\rho\sqrt{1-z}} & \text{for} \quad \frac{r^2}{\rho^2/z} \ll 1 \\ K_0\left(\hat{Q}r\right) & \text{for} \quad \frac{r^2}{\rho^2/z} \gg 1 \end{cases}$

  Hence, in the two limiting cases (pick $z \gg 1-z$)

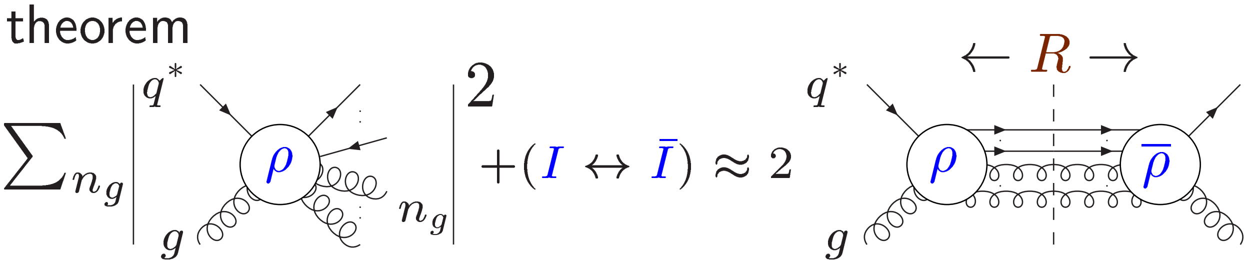
$\frac{r^2}{\langle\rho\rangle^2} \ll 1$: $\left(|\,\Psi_{L,T}\,|^2\,\sigma_{\rm DP}\right)^{(I)} = \mathcal{O}(1)$ but **tiny** (in pQCD: $\left(|\Psi_L|^2+|\Psi_T|^2\right)\sigma_{\rm DP} = \mathcal{O}(1)$)

$$\frac{r^2}{\langle\rho\rangle^2} \gg 1:\ \left(|\,\Psi_{L,T}\,|^2\,\sigma_{\rm DP}\right)^{(I)} \approx |\,\Psi_{L,T}^{\rm pQCD}\,|^2\,\underbrace{\frac{1}{\alpha_s}\,x_{\rm Bj}\,G(x_{\rm Bj},\mu^2)\,\frac{\pi^8}{12}\left(\int_0^\infty d\rho\,D_{\rm lattice}(\rho)\,\rho^5\right)^2}_{\sigma_{\rm DP}^{(I)}\ \text{"saturates"}}$$

Realistic process: $\gamma^* + g \xrightarrow{(I)} n_f\,(q_{\rm R} + \overline{q}_{\rm R}) + $ **gluons**

- I-dynamics in the I-induced subprocess $q^*(q') + g(p) \xrightarrow{(I)} X$ (Off mass shell quark q^* from photon $\gamma \to \overline{q} + q^*$)

- **Valley method** [Balitsky & Yung '86]:
  - Elegant **summation over final-state gluons** via optical theorem



  - Resummation of final-state gluons $\Rightarrow$ valley interaction $\Omega^{I\bar I}_{\rm valley}(\frac{R}{\sqrt{\rho\overline{\rho}}}, \ldots)$ between I's and $\overline{I}$'s:

- Variables: energy: $E = \sqrt{(q'+p)^2}$, virtuality: $Q'^{\,2} = -q'^{\,2}$

$I\bar I$**-interaction** $-1 \leq \Omega^{I\bar I}_{\rm valley} \leq 0$, 2^{nd} **crucial building block** of I-calculus

- Analytic form explicitly known [Khoze & Ringwald '91; Verbaarschot'91]
- I-pert. theory: formally valid for $\sqrt{R^2} \gg \sqrt{\rho\overline{\rho}}$
- Valley approximation might work down to I and $\overline{I}$ almost touching? $\Leftrightarrow \Omega^{I\bar I}_{\rm valley} \approx -\frac{1}{2}$ or $\sqrt{R^2} \approx \sqrt{\rho\overline{\rho}}$ $\Leftarrow$**Lattice!**

- **Strategy** like in "simplest process":
  Start: I-pert. theory $\oplus$ valley method [Ringwald & F. Sch. '98],

  $\rightarrow$ Transformation to colour-dipole picture, continuation to saturation regime by exploiting **lattice** data about $\Omega^{(I\bar{I})}$

- $\sigma_{\rm DP}^{(I)\,\rm gluons}$ involves integration over $\ldots$, E and R ($\Leftarrow$ **gluons!**)
  - In softer regime of interest: $\rho \approx \langle\rho\rangle$
    and $\boldsymbol{R}$-integral dominated by **saddle-point** $R^*(\frac{E}{M_{\rm sph}})$!
  - **Note:** Mass $M_{\rm sph}$ of **QCD-sphaleron** as scale for I-subprocess energy E:
  - $M_{\rm sph}$ $\equiv$ Barrier height separating neighbouring QCD vacua:

$$M_{\rm sph} \approx \frac{3\pi}{4}\,\frac{1}{\alpha_s}\,\frac{1}{\langle\rho\rangle} \approx 2-3\ \text{GeV}$$

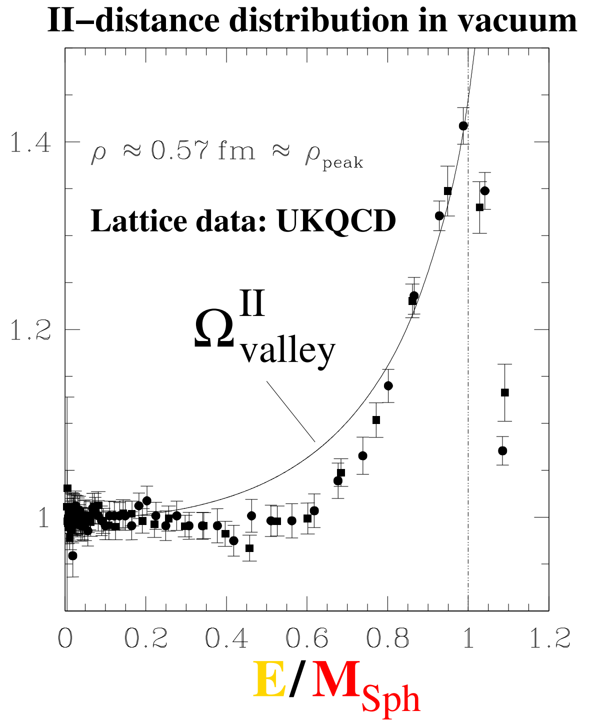
[Ringwald & Schrempp '94; Diakonov & Petrov '94]

- Simulated on the **lattice**: $I\bar{I}$-distance distribution:

$$\frac{d\,n^{(I\bar{I})}}{d^4x_0\,d^4R} \quad\Leftrightarrow\quad \left\langle \exp\left\{-\frac{4\pi}{\alpha_s}\,\boldsymbol{\Omega}^{I\bar{I}}\right\}\right\rangle_{U,\rho,\bar{\rho}}$$

[Ringwald & Schrempp '99]

- Saddle point $\Rightarrow$ $\frac{R}{\langle\rho\rangle} \Leftrightarrow \frac{E}{M_{\rm sph}}$
- **Maximum at $E \approx M_{\rm sph}$**
- $\Omega^{I\bar{I}}_{\rm valley}$ valid right up to the QCD-sphaleron peak!
- For $E \gtrsim M_{\rm sph}$:
  substitute $\Omega^{I\bar{I}}_{\rm valley} \Rightarrow \Omega^{I\bar{I}}_{\rm lattice}$



$$\sigma_{\rm DP}^{(I)\,\rm gluons} \propto \int d\rho \int d\bar\rho\, D_{\rm lattice}(\rho)\rho^5\, D_{\rm lattice}(\bar\rho)\bar\rho^5$$
$$\times \int dE \int d^4R\ \exp\left\{-\frac{4\pi}{\alpha_s}\Omega_{\rm lattice}^{(I\bar I)}(R,\rho,\bar\rho)\right\}[\ldots],$$

- Due to final-state **gluons**:
  E-integral dominated by **QCD-sphaleron** peak, and again

$$\Rightarrow\ \sigma_{\rm DP}^{(I)\,\rm gluons} \propto \left(\int d\rho\ D_{\rm lattice}(\rho)\rho^5\right)^2$$

- such that saturation state of **"color-glass condensate"** $\Leftrightarrow$ **QCD-sphaleron**, a coherent multi-gluon state!

Moreover, it was shown:

- Rise of $\sigma_{\rm DP}^{(I)}(r,\ldots)$ with the area of the dipole ($\propto \pi\, r^2$) flattens out for $r \gtrsim \langle\rho\rangle$

- At larger r, $\sigma_{\rm DP}^{(I)}(r,\ldots)$ reaches a constant value proportional to the area $\pi\langle\rho\rangle^2$ of the background instanton

  Saturation scale $\Leftrightarrow$ intrinsic I-size, $R_s \approx \langle\rho\rangle \approx 0.5$ fm

- I-driven, saturated dipole cross section of hadronic size $\mathcal{O}(\frac{1}{\alpha_s}\pi\langle\rho\rangle^2) = \mathcal{O}(10\ \rm mb)$

- **Open & interesting:**
  $x_{\rm Bj}$-dependence of saturation scale $R_s(x_{\rm Bj})$?

4. Wilson-loop Scattering in an I-Background

- **Consider:** High-energy scattering of partons ($s \gg -t$ or small $x_{\rm Bj}$) from a soft colour field A_μ in non-abelian **eikonal approximation**
  - Soft interaction leaves parton direction $\approx$ unchanged
  - Only non-abelian phase change $\to$ Path-ordered integral along the parton's classical path (**Wilson line**):

$$W(x) = \mathsf{P}\ \exp\left\{-i\,g_s \int_{-\infty}^{+\infty} d\,\lambda\; q^\mu\, A_\mu(\lambda\, q + x_\perp)\right\}$$

  Basic framework of $\bar{q}q$-scattering for $s \gg -t$: [Nachtmann '91]

- $(\bar{q}q)$ colour-dipole $\Leftrightarrow$ **Wilson loop**:

$$\mathcal{W}(\boldsymbol{r}, \boldsymbol{b}; A_\mu) = \frac{1}{N_c}\,\mathrm{tr}\left[W\left(\boldsymbol{b} + \tfrac{\boldsymbol{r}}{2}\right) W^\dagger \left(\boldsymbol{b} - \tfrac{\boldsymbol{r}}{2}\right)\right]$$

  - $\boldsymbol{r}$: transverse size of the colour dipole
  - $\boldsymbol{b}$: transverse distance between dipole and colour field A_μ

- Wilson loop $\mathcal{W}$ is basic object in **BK** evolution equations [Balitsky '98]

- **Simplest scenario only:**

  Proton $\Leftrightarrow$ source of the classical field A_μ
  (cf. also "semi-classical approach" [Buchmüller, Hebecker *et al* '96-'99])

$$\Rightarrow \quad \sigma_{\mathrm{DP}}(r,\ldots) = 2 \int d^2\boldsymbol{b} \, \langle 1 - \mathcal{W}(\boldsymbol{r}, \boldsymbol{b}; A_\mu) \rangle_{A_\mu}$$

- **Here:** Study $\boldsymbol{I}$-contribution to the dipole cross section.

  Analytically calculable $\to$ direct relation between the extension of classical fields and saturation scale

$$A_\mu(x) \to A_\mu^{(I)}(x, \rho, x_0)$$

$$\langle \ldots \rangle_{A_\mu} \to \mathcal{D}A_\mu^{(I)} \to d^4x_0 \, d\rho D(\rho)$$

- **First step**: Calculation of the Wilson loop in the I-background (performed analytically)

$$1 - \mathcal{W}^{(I)}\left(\boldsymbol{r}, \boldsymbol{b}; A_\mu^{(I)}\right)$$

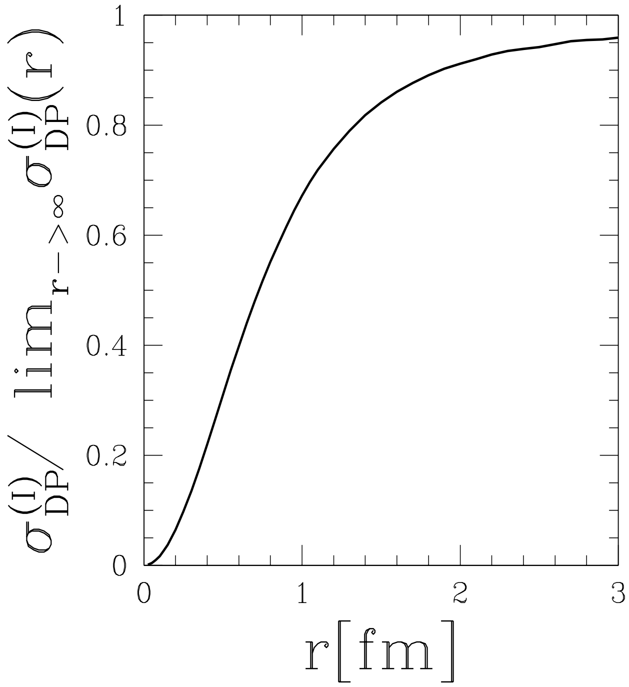
$$= \frac{2}{N_c}\left[1 - \cos\left(\pi\sqrt{\frac{\boldsymbol{x}^2}{\boldsymbol{x}^2+\rho^2}}\right)\cos\left(\pi\sqrt{\frac{\boldsymbol{y}^2}{\boldsymbol{y}^2+\rho^2}}\right) - \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\sqrt{\boldsymbol{x}^2}\sqrt{\boldsymbol{y}^2}}\sin\left(\pi\sqrt{\frac{\boldsymbol{x}^2}{\boldsymbol{x}^2+\rho^2}}\right)\sin\left(\pi\sqrt{\frac{\boldsymbol{y}^2}{\boldsymbol{y}^2+\rho^2}}\right)\right]$$

where $\quad \boldsymbol{x} = \boldsymbol{b} + \frac{\boldsymbol{r}}{2}, \quad \boldsymbol{y} = \boldsymbol{b} - \frac{\boldsymbol{r}}{2}$

- **Second step**: Integration over collective coordinates

$\boxed{\text{Dipole cross section } \sigma_{\rm DP}(r, \ldots):}$

- $\sigma^{(I)}_{\rm DP}(r, \ldots) \propto \langle\rho\rangle^2 f\left(\frac{r}{\langle\rho\rangle}\right)$
  - $r \ll \langle\rho\rangle: \quad \sigma^{(I)}_{\rm DP} \propto r^2$
  - $r \gg \langle\rho\rangle: \quad \sigma^{(I)}_{\rm DP} \propto \langle\rho\rangle^2$



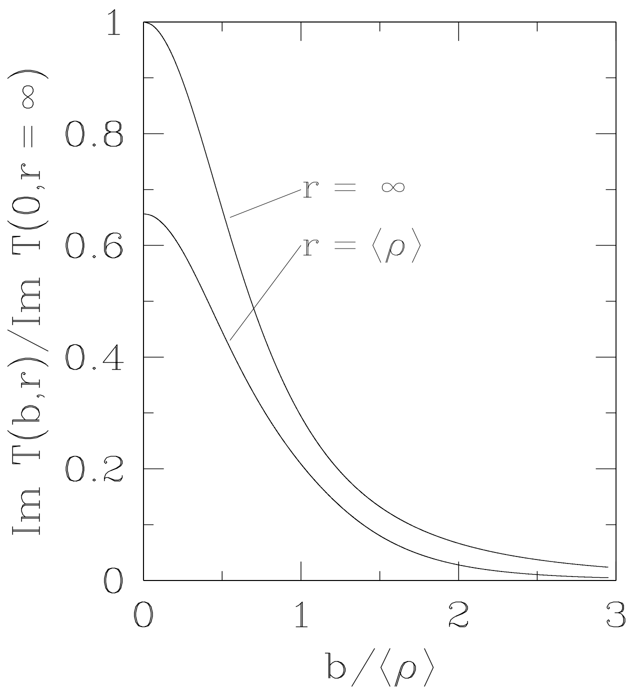
- **Colour transparency** and **saturation**

$\boxed{b\textbf{-profile: } \sigma_{\rm DP} = 2\int d^2b \ \mathrm{Im}\,[T(r, b)]:}$

– Profile is **central**

– Width of b-profile reflects Instanton-size

– **Large** b: $\mathrm{Im}\,[T(r,b)] \propto \frac{1}{b^4}$
general property
e.g.[Golec-Biernat & Stasto, '03]



- **Extensions**: analogous to [Dosch *et al* '87-; Pirner *et al* '02]:
- Dipole-dipole scattering in an I-background
$\rightarrow$ consistent results,. . .

5. Conclusions and Outlook

- **Addressed Questions**:
  - **Saturation** in an (non-perturbative) I-background?
  - Connection between saturation scale and $\boldsymbol{I}$-size $\langle\rho\rangle$?

  studied in the **colour-dipole picture**

- **Two** approaches for investigating $\sigma_{\rm DP}^{(I)}(r,\ldots)$:
  - I-perturbation theory $\oplus$ lattice data
  - Wilson-loop scattering in I-background

- **Similar** and **qualitatively realistic** results:
  - Colour transparency at small r,
  - Rise flattens out for $r \gtrsim \langle\rho\rangle$
    $\rightarrow$ Saturation: $\sigma_{\rm DP}^{(I)} \propto \pi\langle\rho\rangle^2$ with scale $Q_s \sim \frac{1}{\langle\boldsymbol{\rho}\rangle}$
  - **"colour-glass condensate"** $\Leftrightarrow$ **QCD sphaleron**!

- **Outlook**:
  - Study possible reasons for the $x_{\rm Bj}$-dependence of Q_s: Saddle point $R^*(E)$ or **BK**-equations
  - **Interesting:** Detailed phenomenological investigation
  - **By-product:** Exploit experience from continuing I-perturbation theory to softer scales for I-search at **HERA**!
  - Further investigations of conformal invariance of instanton sector and BK-equations