Vorlesung 5: Integralrechnung I

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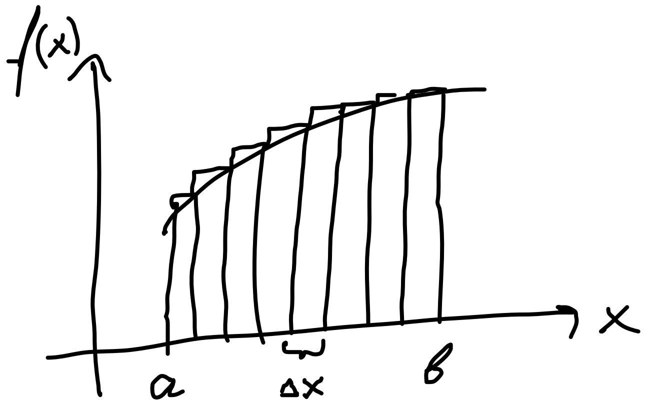
Berechnung Flächeninhalts unterhalb einer Kurve



$$x_k = a + k\cdot\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\cdot\Delta x$$

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ wobei } F'(x) = f(x)$$

$F(x)$ heisst Stammfunktion von $f(x)$

Bemerkungen:

- $F(x)$ nicht eindeutig: $\frac{d}{dx}\left(F(x)+c\right) = \frac{dF(x)}{dx} = f(x) \quad c\in\mathbb{R}$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Liste von Stammfunktionen

$f(x)$	$F(x)$
$x^n,\ n \in \mathbb{Z},\ n \neq -1$	$\frac{1}{n+1} x^{n+1}$
$\frac{1}{x} = x^{-1}$	$\ln x$
$x^{\frac{1}{n}}$	$\frac{n}{n+1} x^{\frac{1}{n}+1}$
$e^{\alpha x},\ \alpha \in \mathbb{R}$	$\frac{1}{\alpha} e^{\alpha x}$
$\sin(\alpha x)$	$-\frac{1}{\alpha}\cos(\alpha x)$
$\cos(\alpha x)$	$\frac{1}{\alpha}\sin(\alpha x)$
$\sinh(\alpha x)$	$\frac{1}{\alpha}\cosh(\alpha x)$
$\cosh(\alpha x)$	$\frac{1}{\alpha}\sinh(\alpha x)$

$$\int x^2 \, dx$$

Beispiel.
$$\int_a^b x^n \, dx = F(b) - F(a) = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

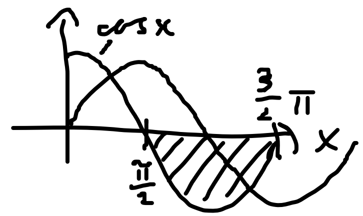
$$\int f(x) \, dx$$

$$= \frac{1}{n+1} x^{n+1} \Big|_a^b$$

$$\int_a^b (x-c)^n \, dx = \frac{1}{n+1}(x-c)^{n+1}\Big|_a^b = \frac{1}{n+1}\left((b-c)^{n+1} - (a-c)^{n+1}\right)$$

$\int_a^b f(x)\,dx$ heisst <u>bestimmtes</u> Integral

$$\left.\begin{aligned} &\int^y f(x)\,dx = F(y)+c \mathrel{\hat=} F(y) \\ &\int f(x)\,dx = F(x) \end{aligned}\right\} \text{ unbestimmtes Integral}$$



$$\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos x\,dx = \sin x \Big|_{\frac{\pi}{2}}^{\frac{3}{2}\pi} = (-1)-(1) = -2$$

negativer Flächeninhalt

Rechenregeln für Integrale

i) Linearität

$$\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx, \quad \alpha, \beta \in \mathbb{R}$$

Beispiel: $\int (\alpha x^n + \beta x^{n-1})\,dx = \frac{\alpha}{n+1} x^{n+1} + \frac{\beta}{n} x^n$

ii) partielle Integration

$$(f\cdot g)' = f'\cdot g + f\cdot g' \Leftrightarrow f'\cdot g = (f\cdot g)' - f\cdot g' \quad \Big| \int \dots dx$$

$$\int f'(x)\,g(x)\,dx = \underbrace{\int (f(x)\,g(x))'\,dx}_{=f(x)\,g(x)} - \int f(x)\,g'(x)\,dx$$

$$\Rightarrow \int f'(x)\,g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx$$

Beispiele.

$f = -\cos x \quad g = x$

$f' = \sin x \quad g' = 1$

- $\int \underset{g}{x} \cdot \underset{f'}{\sin x}\, dx = -x \cdot \cos x + \int 1 \cdot \cos x\, dx = -x \cdot \cos x + \sin x$

  Probe: $\frac{d}{dx}(\sin x - x \cdot \cos x) = \cos x - (1 \cdot \cos x + x \cdot (-\sin x)) = x \cdot \sin x$

- $\int \underset{g}{x^2} \cdot \underset{f'}{e^x}\, dx = x^2 \cdot e^x - 2\int \underset{g}{x} \cdot \underset{f'}{e^x}\, dx = x^2 e^x - 2\left(x \cdot e^x - \int 1 \cdot e^x dx\right) = x^2 e^x - 2xe^x + 2e^x$

  $= (x^2 - 2x + 2)e^x$. Probe: $\frac{d}{dx}\left((x^2-2x+2)e^x\right) = (2x-2)e^x + (x^2-2x+2) \cdot e^x = x^2 e^x$

  Dieses Verfahren funktioniert für alle Polynome $p(x)$, also $\int p(x) e^x dx$

- $\int \underset{f'}{x^n} \underset{g}{\ln x}\, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1}\int \underbrace{x^{n+1} \cdot \frac{1}{x}}_{x^n} dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} \cdot x^{n+1}$

  $= x^{n+1} \cdot \left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right)$ (auch das geht allgemein für Polynome $\int p(x) \ln x\, dx$)

- $\int \cos^2 x\, dx = \int \underset{f'}{\cos x} \cdot \underset{g}{\cos x}\, dx = \sin x \cdot \cos x + \int \underbrace{\sin^2 x}_{1-\cos^2 x} dx = \cos x \sin x + x - \int \cos^2 x\, dx$

  $\int \cos^2 x\, dx = \frac{1}{2}(x + \cos x \cdot \sin x)$

- $\int \underset{g}{e^x} \cdot \underset{f'}{\sin x}\, dx = -e^x \cdot \cos x + \int \underset{g}{e^x} \cdot \underset{f'}{\cos x}\, dx = -e^x \cdot \cos x + e^x \cdot \sin x - \int e^x \cdot \sin x\, dx$

  $\int e^x \cdot \sin x\, dx = \frac{1}{2} e^x \cdot (\sin x - \cos x)$

Substitution:

Kettenregel: $(g(f(x)))' = g'(f(x)) \cdot f'(x) \quad \Big| \int \ldots dx$

$$\int g'(f(x)) \cdot f'(x)\, dx = g(f(x))$$

Das heisst, dass wenn der Integrand eine Ableitung ist, dann kann man die Stammfunktion sofort angeben.

Beispiele: • $\int x \cdot \underbrace{e^{-x^2}}_{g'(f(x))}\, dx = -\frac{1}{2} \int (-2x)\, e^{-x^2} dx = -\frac{1}{2} \cdot e^{-x^2}$ $\qquad g(x) = e^x$, $f(x) = -x^2$

$g'(f(x)) \to f(x)$ eingesetzt in $g'(x)$

• $\int \frac{f'(x)}{f(x)}\, dx = \int (\ln(f(x)))' dx = \ln f(x)$, $\qquad (\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x)$

z. B. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(\cos x)'}{\cos x} = -\ln(\cos x)$

formale Vereinfachung durch Substitutionsregel

$f(x) = z \qquad \frac{dz}{dx} = f'(x) \Rightarrow dz = f'(x)\, dx \qquad g' = h$

$\to \int g'(f(x)) f'(x)\, dx = \int h(z) \frac{dz}{dx} dx = \int h(z)\, dz$

Beispiel: $\int x \cdot e^{-x^2} dx$. Substitution: $z = x^2$, $h = e^{-z}$, $\frac{dz}{dx} = 2x \Leftrightarrow \frac{1}{2} dz = x\, dx$

$\Rightarrow \int x \cdot e^{-x^2} dx = \frac{1}{2} \int e^{-z}\, dz = \frac{1}{2}(-e^{-z}) = -\frac{1}{2} e^{-x^2}$

- $\int \sin x \cdot \cos^n x \, dx = -\int z^n \, dz = -\frac{1}{n+1} \cdot z^{n+1} = -\frac{1}{n+1} \cdot \cos^{n+1} x$

  $z = \cos x$, $h(z) = z^n$, $\frac{dz}{dx} = -\sin x \Leftrightarrow \sin x \, dx = -dz$