Nachtrag Übungsblatt 10

3.) i) $$\int_2^3 \int_x^{x^2} (x+2y)\,dy\,dx = \int_2^3 \left[xy+y^2\right]_x^{x^2} dx =$$

$$= \int_2^3 \left(x^3+x^4-x^2-x^2\right)dx = \int_2^3 \left(x^3+x^4-2x^2\right)dx$$

$$= \left[\frac{x^4}{4}+\frac{x^5}{5}-\frac{2}{3}x^3\right]_2^3 = \frac{1}{60}\left[15(3^4-2^4)+12(3^5-2^5) - 40(3^3-2^3)\right]$$

$$= \frac{2747}{60}$$

ii) $$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dy\,dx$$

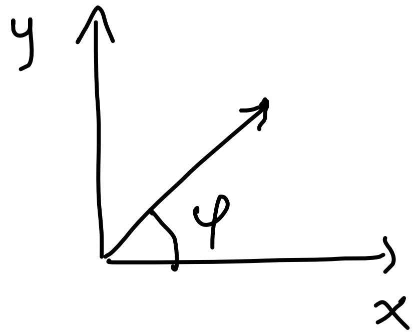
$$\int e^{-x^2}dx = \mathrm{Erf}(x)$$

in Polarkoordinaten:

$$x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\varphi$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,d\varphi\,dr =$$



$$= 2\pi\int_0^{\infty} e^{-r^2}\,r\,dr =$$

$$r' := r^2$$

$$dr' = 2r\,dr$$

$$\pi\int_0^{\infty} e^{-r'}\,dr' = -\pi\, e^{-r'}\Big|_0^{\infty}$$

$$= \pi$$

(iii) $$\int_0^1\int_0^{1-x}\int_0^{1-x-y}(2x+y+z)\,dz\,dy\,dx =$$

$$= \int_0^1\int_0^{1-x}\left[(2x+y)z + \frac{z^2}{2}\right]_0^{1-x-y} dy\,dx =$$

$$= \int_0^1\int_0^{1-x}\underbrace{\left[(2x+y)(1-x-y) + \frac{(1-x-y)^2}{2}\right]}_{\frac{1}{2}(1+2x-3x^2-4xy-y^2)} dy\,dx$$

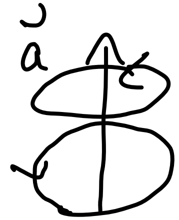
$$= \frac{1}{2}\int_0^1\left[(1+2x-3x^2)y - 2xy^2 - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1\left[(1+2x-3x^2)(1-x) - 2x(1-x)^2 - \frac{(1-x)^3}{3}\right] dx = \frac{1}{6}$$

4.) $\vec{r} = r\vec{e}_r \qquad \vec{e}_r = \frac{\vec{r}}{r} \qquad r = \sqrt{\vec{r}^2} = \sqrt{x^2+y^2+z^2}$

$$\vec{r} = x\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$$

$$\vec{a} = a_x\,\vec{e}_x + a_y\,\vec{e}_y + a_z\,\vec{e}_z$$



i) $\vec{F}(\vec{r}) = \vec{a} \times \vec{r}$

Divergenz: $\vec{\nabla} \cdot \vec{F}(\vec{r}) = \vec{\nabla} \cdot (\vec{a} \times \vec{r}) =$

$$= \vec{\nabla} \cdot \left[(a_y z - a_z y)\vec{e}_x + (a_z x - a_x z)\,\vec{e}_y + (a_x y - a_y x)\vec{e}_z\right]$$

$$= \frac{\partial}{\partial x}(a_y z - a_z y) + \dots = 0$$

Rotation: $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \left[(a_y z - a_z y)\,\vec{e}_x + (a_z x - a_x z)\,\vec{e}_y + (a_x y - a_y x)\vec{e}_z\right]$

$$= \qquad (\breve{\nabla}\times\breve{F})_x = \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y$$

$$= \left[\frac{\partial}{\partial y}(a_x y - a_y x) - \frac{\partial}{\partial z}(a_z x - a_x z)\right]\breve{e}_x$$

$$+ \left[\frac{\partial}{\partial z}(a_y z - a_z y) - \frac{\partial}{\partial x}(a_x y - a_y x)\right]\breve{e}_y$$

$$+ \left[\frac{\partial}{\partial x}(a_z x - a_x z) - \frac{\partial}{\partial y}(a_y z - a_z y)\right]\breve{e}_z$$

$$= 2a_x\breve{e}_x + 2a_y\breve{e}_y + 2a_z\breve{e}_z = 2\vec{a}$$

(ii) $\breve{F}(\breve{r}) = (2x+y)\,\breve{e}_x + (x+2yz)\,\breve{e}_y + (y^2+2z)\,\breve{e}_z$

Divergenz: $\breve{\nabla}\cdot\breve{F} = 2 + 2z + 2 = 4 + 2z$

$$\tilde{\nabla} \times \tilde{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\tilde{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\tilde{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\tilde{e}_z$$

$$= (2y - 2y)\,\tilde{e}_x + 0 \cdot \tilde{e}_y + (1 - 1)\,\tilde{e}_z = \vec{0}$$

(iii) $\tilde{F}(\tilde{r}) = \dfrac{\tilde{r}}{r^2} = \dfrac{1}{r^2}(x, y, z)$ $\qquad r^2 = x^2 + y^2 + z^2$

Divergenz: $\tilde{\nabla} \cdot \tilde{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z} =$

$$= \frac{3}{r^2} + x \frac{\partial}{\partial x}\frac{1}{r^2} + y \frac{\partial}{\partial y}\frac{1}{r^2} + z \frac{\partial}{\partial z}\frac{1}{r^2}$$

$$= \frac{3}{r^2} - \frac{2}{r^4}\left(x^2 + y^2 + z^2\right) = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$$\Rightarrow \tilde{\nabla} \cdot \tilde{F} = \frac{1}{r^2} \quad ; \quad \tilde{\nabla} \times \tilde{F} = 0$$

$$\frac{\partial}{\partial x}\frac{1}{r^2} = -\frac{2x}{(x^2+y^2+z^2)^2} = -\frac{2x}{r^4}$$

4b) : ii) und iii) sind Gradientenfelder da $\underset{\sim}{\nabla} \times \underset{\sim}{F} = 0$

ii) : $\underset{\sim}{F} = \underset{\sim}{\nabla} f$

<u>Weg 1:</u>

$$\frac{\partial f}{\partial x} = F_x = 2x + y \quad \Rightarrow \quad f = x^2 + yx + C_1(y, z)$$

$$\frac{\partial f}{\partial y} = F_y = x + 2yz \quad \Rightarrow \quad f = xy + y^2 z + C_2(x, z)$$

$$\frac{\partial f}{\partial z} = F_z = y^2 + 2z \quad \Rightarrow \quad f = y^2 z + z^2 + C_3(x, y)$$

$$\Rightarrow \; f(x, y, z) = (x^2 + z^2 + xy + y^2 z) \quad \text{für } \underset{\sim}{F} = \underset{\sim}{\nabla} f$$

$$f(x, y, z) = -(x^2 + z^2 + xy + y^2 z) \quad \text{für } \underset{\sim}{F} = -\underset{\sim}{\nabla} f$$

Weg 2: Falls $\vec{\nabla} \times \vec{F} = 0$ und damit $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$ wegunabhängig ist, kann man $f(\vec{r})$ bestimmen, indem man zwischen $\vec{r}_1$ und $\vec{r}_2$ den direkten Weg wählt

Für $\vec{r}_1 = \vec{0}$

$$f(\vec{r}) = \text{const.} + \int_0^1 \vec{r} \cdot \vec{F}(t\vec{r})\, dt =$$

$$= \text{const.} + \int_0^1 \left[x F_x(t\vec{r}) + y F_x(t\vec{r}) + z F_z(t\vec{r}) \right] dt$$

Für ii) führt das auf:

$$f(\vec{r}) = \int_0^1 \left[x(2tx + ty) + y(tx + 2t^2 yz) + z(t^2 y^2 + 2tz) \right] dt$$

$$= \int_0^1 \left[2x^2 t + 2xy\, t + 3y^2 z t^2 + 2z^2 t \right] dt =$$

$$= x^2 + xy + y^2 z + z^2$$

iii) $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{r^2}$ $\quad |\vec{F}(\vec{r})| = \dfrac{1}{r}$

$f(\vec{r}) = f(r)$ $\quad \dfrac{\partial f}{\partial r} = \dfrac{1}{r}$ $\Rightarrow$ $f(r) = \ln r$

$$\frac{\partial}{\partial x} f(r) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{r}\underbrace{\frac{x}{r}}_{\vec{e}_x} = \frac{x}{r^2}$$

$$\Rightarrow \vec{\nabla} f = \frac{\vec{r}}{r^2}$$

1d)

(i) $\breve{\nabla} f(r) = \frac{\partial f}{\partial r} \breve{e}_r$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} = \frac{x}{r} \frac{\partial f}{\partial r} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial f}{\partial y} = \frac{y}{r} \frac{\partial f}{\partial r}$$

$$\frac{\partial f}{\partial z} = \frac{z}{r} \frac{\partial f}{\partial r}$$

$$\Rightarrow \breve{\nabla} f = \frac{\partial f}{\partial r} \frac{\breve{r}}{r} = \frac{\partial f}{\partial r} \breve{e}_r$$

(ii) $\breve{\nabla} \frac{1}{|\breve{r} - \breve{a}|} = \left(\frac{\partial}{\partial x} \frac{1}{|\breve{r} - \breve{a}|}, \frac{\partial}{\partial y} \frac{1}{|\breve{r} - \breve{a}|}, \frac{\partial}{\partial z} \frac{1}{|\breve{r} - \breve{a}|} \right)$

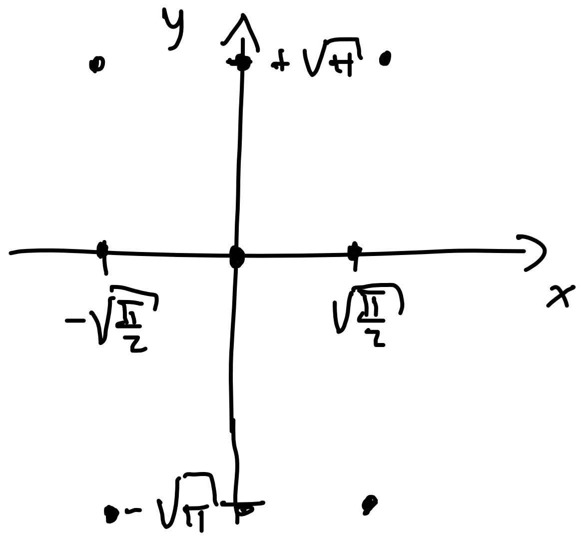
$$\frac{\partial}{\partial x} \frac{1}{|\breve{r} - \breve{a}|} = - \frac{x - a_x}{\left[(x-a_x)^2 + (y-a_y)^2 + (z-a_z)^2\right]^{3/2}} \quad \Bigg| \quad |\breve{r} - \breve{a}| = \sqrt{(x-a_x)^2 + (y-a_y)^2 + (z-a_z)^2}$$

$$\Rightarrow \quad \tilde{\nabla} \frac{1}{|\tilde{r}-\tilde{a}|} = \frac{\tilde{a}-\tilde{r}}{|\tilde{a}-\tilde{r}|^3} = \frac{\tilde{a}-\tilde{r}}{|\tilde{r}-\tilde{a}|^3}$$

2.) a) $\frac{\partial f}{\partial x} = 2x\cos(x^2)$; $\frac{\partial f}{\partial y} = -2y\sin(y^2)$

$\frac{\partial f}{\partial x} = 0$ bei $x=0$ oder $x = \pm\sqrt{\frac{\pi}{2}}$

$\frac{\partial f}{\partial y} = 0$ bei $y=0$ oder $y = \pm\sqrt{\pi}$



2b) $\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2) - 4x^2\sin(x^2)$; $\frac{\partial^2 f}{\partial y^2} = -2\sin(y^2) - 4y^2\cos(y^2)$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\Rightarrow \quad H = \begin{pmatrix} 2\cos(x^2) - 4x^2\sin(x^2) & 0 \\ 0 & -2\sin(y^2) - 4y^2\cos(y^2) \end{pmatrix}$$

$(0,0)$ $\quad H = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ unbestimmt

$\left(\pm\sqrt{\frac{\pi}{2}}, 0\right)$ $\quad H = \begin{pmatrix} -2\pi & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ unbestimmt

$(0, \pm\sqrt{\pi})$ $\quad H = \begin{pmatrix} 2 & 0 \\ 0 & 4\pi \end{pmatrix} \rightarrow$ Minima

$\left(\pm\sqrt{\frac{\pi}{2}}, \pm\sqrt{\pi}\right)$ $\quad H = \begin{pmatrix} -2\pi & 0 \\ 0 & 4\pi \end{pmatrix} \rightarrow$ Sattelpunkt