Divergenz $\vec{\nabla} \cdot \vec{A}$ als Quellenfeld

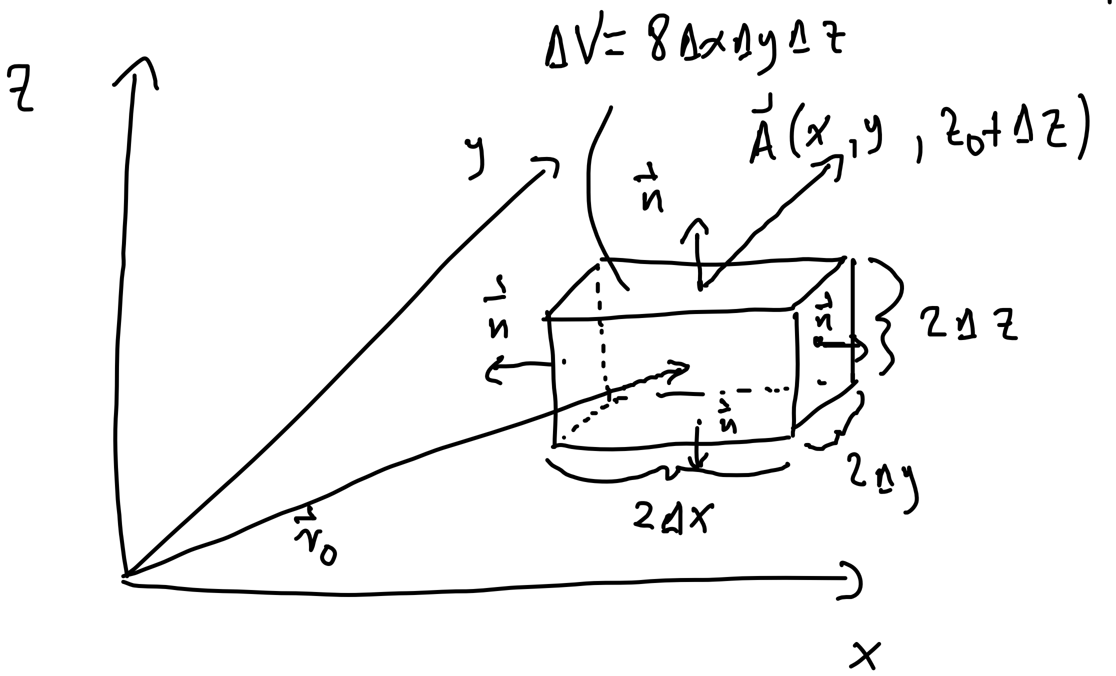
Es gilt bei einem Punkt $\vec{r}_0 = (x_0, y_0, z_0)$

$$\vec{\nabla} \cdot \vec{A}(\vec{r}_0) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\partial(\Delta V)} \vec{A} \cdot d\vec{F}$$

Beweis

Quader um $\vec{r}_0$ mit Seitenlängen $2\Delta x, 2\Delta y, 2\Delta z$

$\vec{A}(x,y,z)$



$d\vec{F} = \vec{n}\,dx\,dy$

$$\iint_{\partial(\Delta V)} \vec{A}\cdot d\vec{F} = \underbrace{\iint \left[A_z(x,y,z_0+\Delta z) - A_z(x,y,z_0-\Delta z)\right] dx\,dy}$$

$= \int \vec{A}\cdot d\vec{F}$ über obere und untere Begrenzungsfläche

$$+\iint \left[A_y(x, y_0+\Delta y, z) - A_y(x, y_0-\Delta y, z) \right] dx\,dz$$

$$+\iint \left[A_x(x_0+\Delta x, y, z) - A_x(x_0-\Delta x, y, z) \right] dy\,dz$$

Betrachte das erste Integral:

$$A_z(x, y, z_0 \pm \Delta z) = A_z(x, y, z_0) \pm \frac{\partial A_z}{\partial z}(x, y, z_0)\,\Delta z + \frac{1}{2}\frac{\partial^2 A_z}{\partial z^2}(x, y, z_0)\,\Delta z^2 + \ldots O(\Delta z^3)$$

Taylorentwicklung nach z um den Punkt $z = z_0$

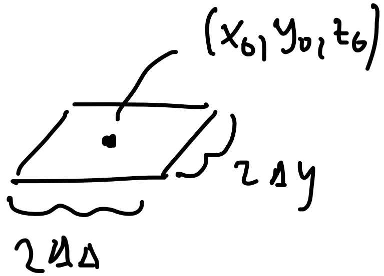
$$A_z(x, y, z_0+\Delta z) - A_z(x, y, z_0-\Delta z) = 2\frac{\partial A_z}{\partial z}(x, y, z_0)\,\Delta z + O(\Delta z^3)$$

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}(x_0)\,(x-x_0) + \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}(x_0)\,(x-x_0)^2 + \dots$$

$$\frac{1}{\Delta V}\iint \left[A_z(x,y,z_0+\Delta z) - A_z(x,y,z_0-\Delta z)\right] dx\,dy =$$

$$= \frac{2\Delta z}{8\Delta x\Delta y\Delta z}\left[\iint \frac{\partial A_z}{\partial z}(x,y,z_0)\,dx\,dy + O(\Delta z^2)\right]$$

$$= \frac{1}{4\Delta x\Delta y}\iint \frac{\partial A_z}{\partial z}(x,y,z_0)\,dx\,dy$$



$$\to \frac{1}{4\Delta x\Delta y}\,\frac{\partial A_z}{\partial z}(x_0,y_0,z_0)\,2\Delta x\,2\Delta y \qquad \text{für } \Delta x, \Delta y \to 0$$

$$= \frac{\partial A_z}{\partial z}(x_0, y_0, z_0)$$

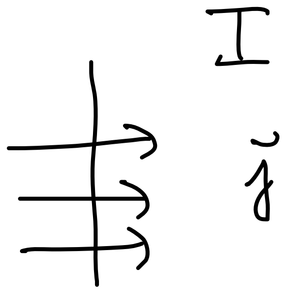
$$\Rightarrow \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{1}{\underbrace{8 \Delta x \Delta y \Delta z}_{\Delta V}} \oiint \vec{A} \cdot d\vec{F} = \frac{\partial A_x}{\partial x}(x_0, y_0, z_0) + \frac{\partial A_y}{\partial y}(x_0, y_0, z_0) + \frac{\partial A_z}{\partial z}(x_0, y_0, z_0) = \vec{\nabla} \cdot \vec{A}(x_0, y_0, z_0)$$

damit ist $\displaystyle \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\partial(\Delta V)} \vec{A} \cdot d\vec{F} = \vec{\nabla} \cdot \vec{A}(\vec{r}_0)$

Einschub: Kontinuitätsgleichung für elektrische Ladung

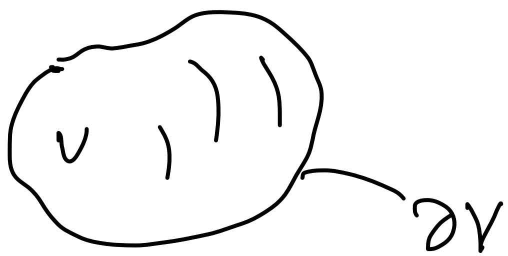
ρ_e = Ladungsdichte

$\vec{j}$ = Stromdichte



$\vec{j} = \dfrac{\text{Strom}}{\text{Fläche}}$

$$\boxed{\frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0}$$



$$Q = \int \rho_e \, dV \qquad \frac{\partial Q}{\partial t} \neq 0$$

$$\iiint \frac{\partial \rho_e}{\partial t} \, dV = -\iiint \vec{\nabla} \cdot \vec{j} \, dV$$

$$\shortparallel$$

$$\frac{\partial Q}{\partial t} = -\iint_{\partial V} \vec{j} \cdot d\vec{F} = -I =$$

− Gesamtstrom durch
Begrenzungsfläche ∂V

Rotation als Wirbelfeld

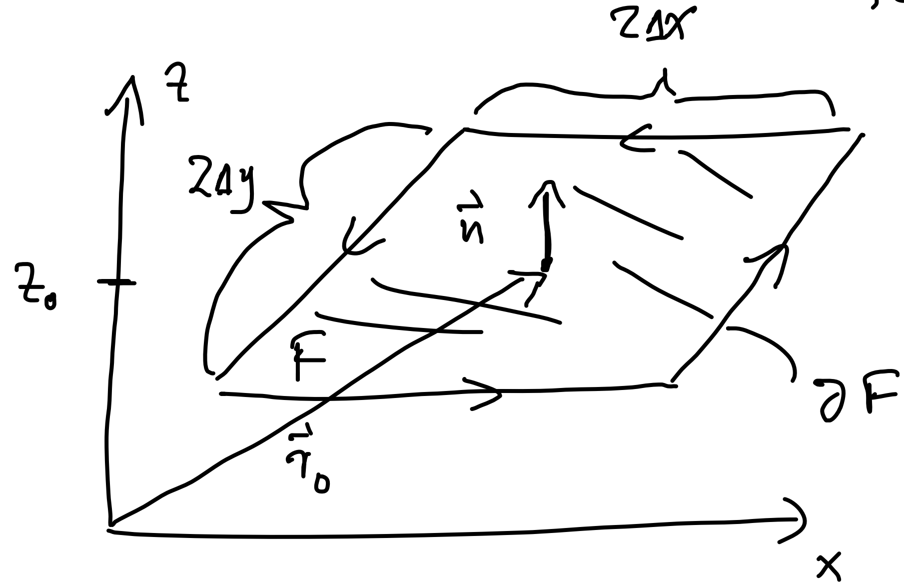
Es gilt:

$$\vec n\cdot\left(\vec\nabla\times\vec A\right)=\lim_{\Delta F\to 0}\frac{1}{\Delta F}\oint\vec A\cdot d\vec r$$

Normale der Fläche ΔF

Orientierung von Umlaufsinn und Normalenvektor sind durch Rechtsschraubenregel bestimmt

Beweis:



(x_0, y_0, z_0)

$$\oint_{\partial F} \breve{A} \cdot d\breve{r} = \int_{x_0-\Delta x}^{x_0+\Delta x} \left[A_x(x, y_0-\Delta y, z_0) - A_x(x, y_0+\Delta y, z_0) \right] dx$$

$$+ \int_{y_0-\Delta y}^{y_0+\Delta y} \left[A_y(x_0+\Delta x, y, z_0) - A_y(x_0-\Delta x, y, z_0) \right] dy$$

$$= -\frac{\partial A_x}{\partial y}(x_0, y_0, z_0)\, 2\Delta y \int_{x_0-\Delta x}^{x_0+\Delta x} dx + \frac{\partial A_y}{\partial x}(x_0, y_0, z_0)\, 2\Delta x \int_{y_0-\Delta y}^{y_0+\Delta y} dy =$$

lim $\Delta x, \Delta y \to 0$

$$= 4\Delta x \Delta y \left[\frac{\partial A_y}{\partial x}(x_0, y_0, z_0) - \frac{\partial A_x}{\partial y}(x_0, y_0, z_0) \right]$$

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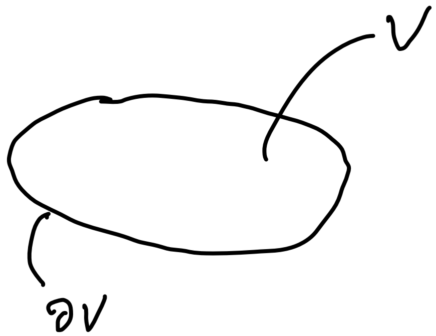
da $\quad A_x(x, y_0 \pm \Delta y, z_0) = A_x(x, y_0, z_0) \pm \frac{\partial A_x}{\partial y}(x, y_0, z_0)\, \Delta y + O(\Delta y^2)$

$$\Rightarrow \lim_{\Delta F \to 0} \frac{1}{\Delta F} \oint \vec{A} \cdot d\vec{r} = \frac{\partial A_y}{\partial x}(\vec{r}_0) - \frac{\partial A_x}{\partial y}(\vec{r}_0) =$$

$$= \left[\vec{\nabla} \times \vec{A}(\vec{r}_0)\right]_z = (\vec{\nabla} \times \vec{A}) \cdot \vec{n}$$

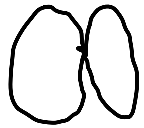
Satz von Gauß



$$\iint_{\partial V} \vec{A} \cdot d\vec{F} = \iiint \vec{\nabla} \cdot \vec{A} \, dV$$

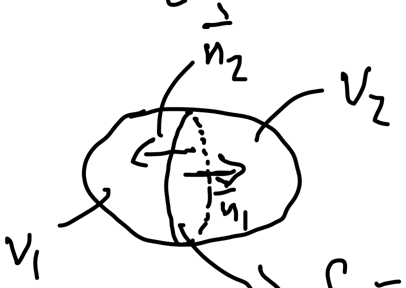
$$V = \bigcup_{i=1}^{N} V_i = V_1 \cup V_2 \cup \ldots \cup V_N$$ mit $V_i \cap V_j = \emptyset$ für $i \neq j$

$$\iiint \vec{\nabla}\cdot\vec{A}\,dV = \sum_{i=1}^{N} \iiint_{V_i} \vec{\nabla}\cdot\vec{A}\,dV =$$



$$= \sum_{i=1}^{N} V_i \underbrace{\frac{1}{V_i}\iiint_{V_i} \vec{\nabla}\cdot\vec{A}\,dV}_{\text{für } V_i \to 0 \;\to\; \vec{\nabla}\cdot\vec{A}} = \sum_{i=1}^{N} V_i \left(\vec{\nabla}\cdot\vec{A}\right)_i =$$

$$= \sum_{i=1}^{N} \oiint_{\partial V_i} \vec{A}\cdot d\vec{F} = \oiint_{\partial V} \vec{A}\cdot d\vec{F}$$



$V = V_1 \cup V_2$

S = Schnittfläche der Teilvolumina