Extrema mit Nebenbedingungen

Methode der Lagrange-Multiplikators

Problem: Extremiere eine Funktion $f(x,y,z)$ unter einer Nebenbedingung $\phi(x,y,z) = 0$

$$F(x,y,z,\lambda) := f(x,y,z) + \lambda\, \phi(x,y,z)$$

$$\frac{\partial F}{\partial \lambda} = \phi(x,y,z) = 0$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

Für unser Quader-Beispiel hat man dann

$$F(x,y,z,\lambda) = V^2(x,y,z) + \lambda\left(x^2+y^2+z^2-R^2\right)$$
$$= 64x^2y^2z^2 + \lambda\left(x^2+y^2+z^2-R^2\right)$$

$$\frac{\partial F}{\partial x} = 128xy^2z^2 + 2\lambda x = 0 \quad \Rightarrow \quad \lambda = -64y^2z^2$$

$$\frac{\partial F}{\partial y} = 128x^2yz^2 + 2\lambda y = 0 \quad \Rightarrow \quad \lambda = -64x^2z^2$$

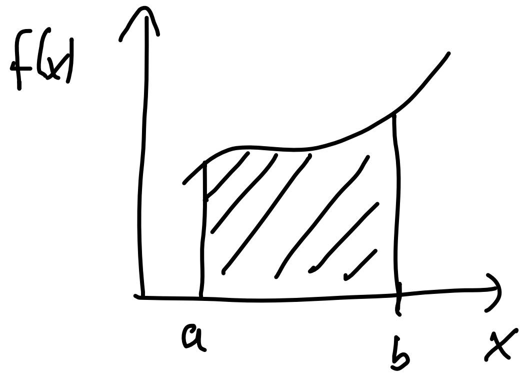
$$\frac{\partial F}{\partial z} = 128x^2y^2z + 2\lambda z = 0 \quad \Rightarrow \quad \lambda = -64x^2y^2$$

$$\Rightarrow \quad x^2 = y^2 = z^2 \qquad \Rightarrow \quad 3x^2 = 3y^2 = 3z^2 = R^2$$

$$\Rightarrow \quad x = \frac{R}{\sqrt{3}} = y = z$$

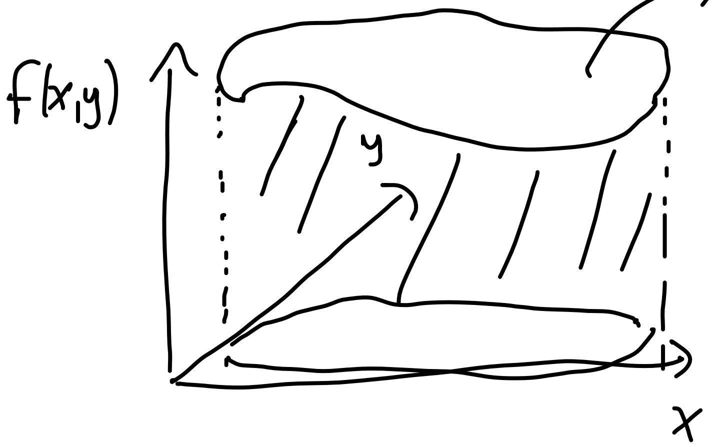
Mehrfachintegrale

Einfacher Integral

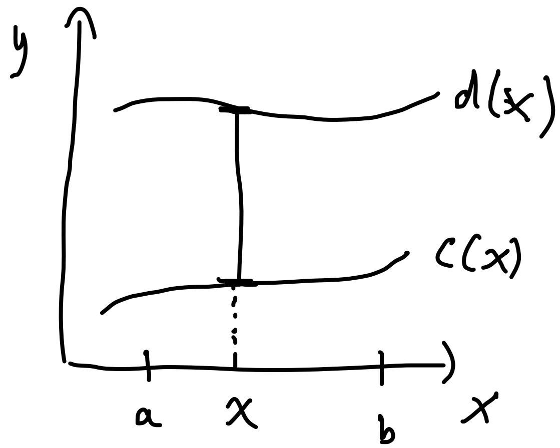


$$I = \int_a^b f(x)\,dx$$

Integral über 2 Variablen



die von $f(x,y)$ definierte Fläche



$$I = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx = \int_a^b F(x,y) \Big|_{y=c(x)}^{y=d(x)} dx$$

$F(x,y)$ $\qquad \dfrac{\partial F}{\partial y}(x,y) = f(x,y)$

$$I = \int_a^b \left[F(x, d(x)) - F(x, c(x)) \right] dx = G(x) \Big|_a^b = G(b) - G(a)$$

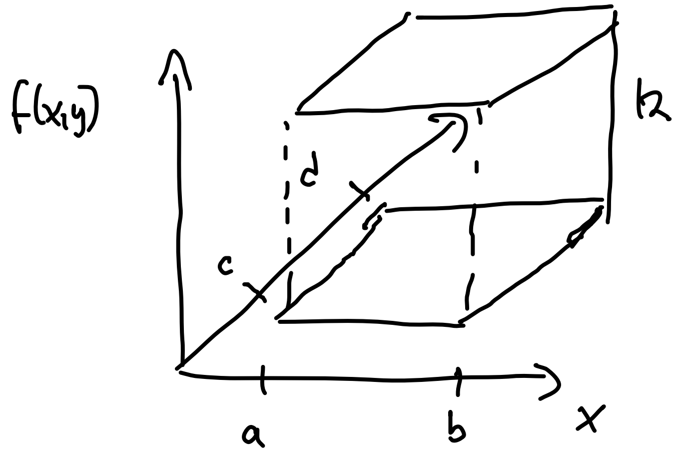
Stammfunktion $G(x)$ mit $\dfrac{dG}{dx} = F(x, d(x)) - F(x, c(x))$

Beispiel 1:

$f(x,y) = k = \text{const.}$ integriert über ein Rechteck

$x \in [a,b]$; $y \in [c,d]$ also $c(x) = c = \text{const.}$

$d(x) = d = \text{const.}$



$V = (b-a)(d-c)\,k$

$$I = \int_a^b \left(\int_c^d k \, dy \right) dx = \int_a^b k y \Big|_c^d \, dx = \int_a^b k(d-c)\, dx$$

$$= k(d-c)\, x \Big|_a^b = k \underbrace{(d-c)(b-a)}_{\text{Fläche des Definitionsbereichs}}$$

$\longrightarrow$ Volumen des Quaders mit Höhe k über Grundfläche $(d-c)(b-a)$

Beispiel 2: Integriere $f(x,y) = 2xy$ über die gleiche Grundfläche: $a \le x \le b$; $c \le y \le d$

$$I = \int_a^b \left(\int_c^d 2xy \, dy \right) dx = \int_a^b 2x \left(\int_c^d y \, dy \right) dx =$$

$$= \int_a^b 2x \left(\left. \frac{y^2}{2} \right|_c^d \right) dx = \int_a^b x (d^2 - c^2) \, dx = (d^2 - c^2) \int_a^b x \, dx$$

$$= (d^2 - c^2) \frac{1}{2} (b^2 - a^2)$$

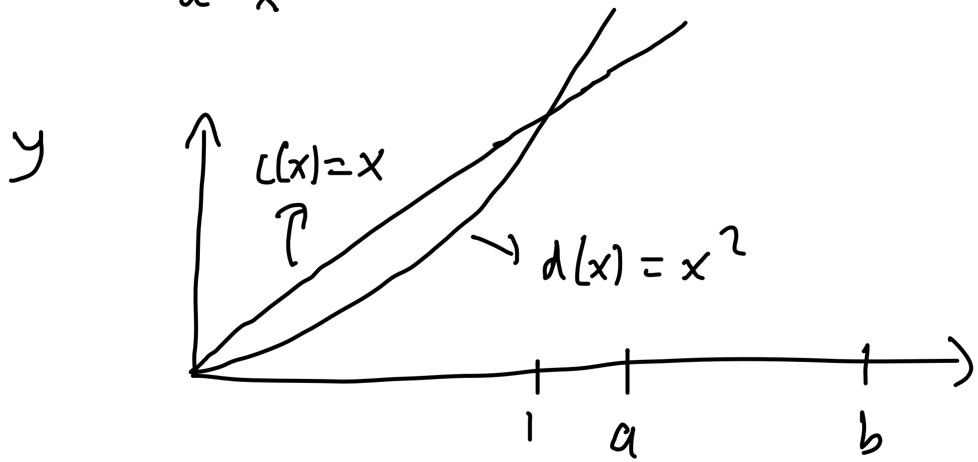
Vertausche Integrationsreihenfolge:

$$I' = \int_c^d \left(\int_a^b 2xy \, dx \right) dy = \int_c^d 2y \left(\int_a^b x \, dx \right) dy =$$

$$= \int_c^d y (b^2 - a^2) \, dy = \frac{1}{2} (b^2 - a^2)(d^2 - c^2) = I$$

Beispiel 3: $f(x,y) = x + 2y$ mit $c(x) = x$ und $d(x) = x^2$

$$I = \int_a^b \int_x^{x^2} (x+2y)\,dy\,dx$$

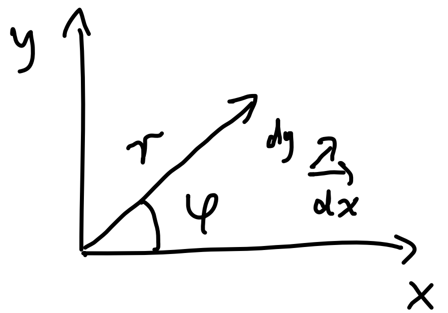


$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$I = \int_a^b \left(\int_{y=x}^{y=x^2} (x+2y)\,dy \right) dx = \int_a^b \left(xy + y^2 \right) \Big|_{y=x}^{y=x^2} dx =$$

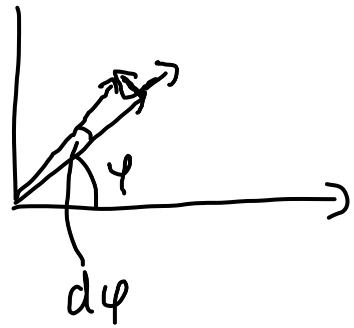
$$= \int_a^b \left(x^3 + x^4 - x^2 - x^2\right) dx = \int_a^b \left(x^3 + x^4 - 2x^2\right) dx$$

$$= \left(\frac{x^4}{4} + \frac{x^5}{5} - \frac{2}{3}x^3\right)\Bigg|_a^b = \frac{1}{4}\left(b^4 - a^4\right) + \frac{1}{5}\left(b^5 - a^5\right) - \frac{2}{3}\left(b^3 - a^3\right)$$



$$(x, y) \to (r, \varphi)$$

$$dF = dx\,dy = r\,dr\,d\varphi$$



Manchmal ist es einfacher die Integration in einem anderen (als kartesisch) Koordinatensystem durchzuführen

$$(x,y) \to (u,v)$$

$$\iint f(x,y)\,dx\,dy \neq \iint f(u,v)\,du\,dv \qquad \text{i.a. nicht}$$

$$\iint f(x,y)\,dx\,dy = \iint f(u,v)\,J(u,v)\,du\,dv$$

$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

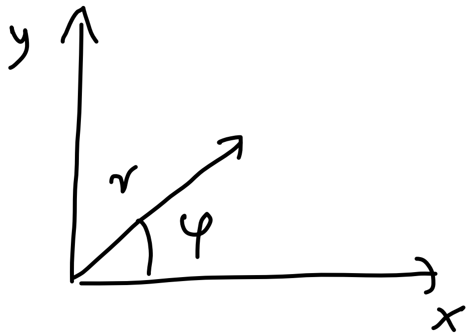
Funktionaldeterminante

Jacobi-Determinante

$$\iint \dots \int f(x_1, \dots, x_n)\,dx_1 \dots dx_n = \iint \dots \int f(y_1, \dots, y_n)\,J(y_1, \dots, y_n)\,dy_1 \dots dy_n$$

$$J(y_1, \ldots, y_n) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

Beispiel:



Polarkoordinaten:

$$x = r\cos\varphi \qquad y = r\sin\varphi$$

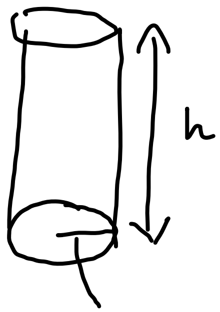
$$\frac{\partial x}{\partial r} = \cos\varphi \; ; \quad \frac{\partial x}{\partial \varphi} = -r\sin\varphi$$

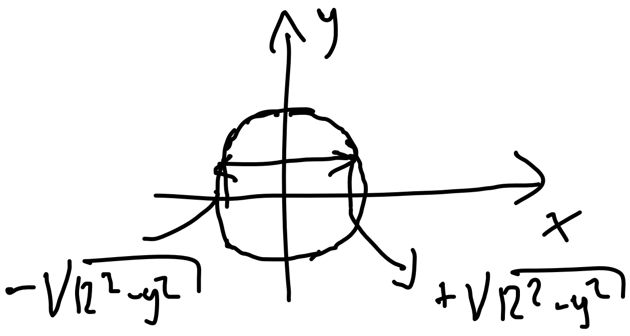
$$\frac{\partial y}{\partial r} = \sin\varphi \; ; \quad \frac{\partial y}{\partial \varphi} = r\cos\varphi$$

$$\Rightarrow J = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r(\cos^2\varphi + \sin^2\varphi) = r$$

$$\iint f(x,y)\,dx\,dy = \iint f(r,\varphi)\, r\, dr\, d\varphi$$

Beispiel 1: Volumen eines Kreiskegels mit Höhe h und Radius R



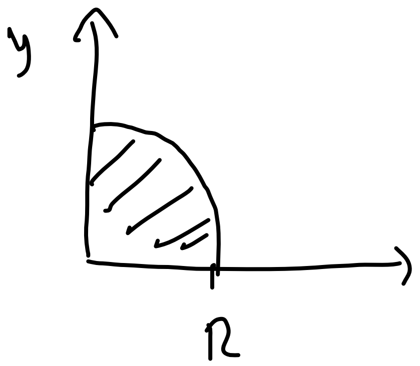


Mit $f(x,y) = h = \text{const.}$

$$V = \iint\limits_{x^2+y^2 \le R^2} h\, dx\,dy = h \int_{-R}^{R} \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} dx\, dy =$$

$$V = \int_{r=0}^{r=R} \int_{\varphi=0}^{2\pi} h\, r\, d\varphi\, dr = 2\pi h \int_{r=0}^{R} r\, dr = \pi R^2 h$$

Beispiel 2: Integriere $f(x,y) = x^2 + y^2$ über ein Kreissegment $x \geq 0,\ y \geq 0$, $x^2 + y^2 \leq R^2$



$$f(x,y) = r^2 = f(r, \cancel{\varphi})$$

$$I = \int_{r=0}^{R} \int_{\varphi=0}^{\pi/2} r^2 \, r \, d\varphi \, dr = \int_{r=0}^{R} r^3 \frac{\pi}{2} \, dr = \frac{\pi}{8} R^4$$

$$I = \iint_{\substack{x,y \geq 0 \\ x^2+y^2 \leq R^2}} (x^2+y^2) \, dx \, dy$$

Analog definiert man Dreifachintegrale

Integration von $f(x,y,z)$ über ein Volumen

$$\int_a^b \int_{c(x)}^{d(x)} \int_{e(x,y)}^{g(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Beispiel 1: $f(x,y,z) = h = \text{const.}$ integriert über einen Quader

$$x \in [a,b]\,,\; y \in [c,d]\,,\; z \in [e,g]$$

$$\int_a^b \int_c^d \int_e^g h\,dz\,dy\,dx = h \int_a^b \int_c^d z\Big|_e^g\,dy\,dx = h(g-e)\int_a^b y\Big|_c^d\,dx$$

$$= h(g-e)(d-c)(b-a) = hV$$

V = Volumen des Quaders

Beispiel 2:

$$\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=-1}^{+1}(x^2+y^2+z^2)\,dz\,dy\,dx =$$

$$= \int_0^2\int_0^1\left(x^2z+y^2z+\frac{z^3}{3}\right)\Big|_{-1}^{+1} dy\,dx = \int_0^2\int_0^1\left(2x^2+2y^2+\frac{2}{3}\right)dy\,dx$$

$$= \int_0^2\left(2x^2y+\frac{2}{3}y^3+\frac{2}{3}y\right)\Big|_0^1 dx = \int_0^2\left(2x^2+\frac{4}{3}\right)dx$$

$$= \left(\frac{2}{3}x^3+\frac{4}{3}x\right)\Big|_0^2 = \frac{2}{3}\cdot 8+\frac{8}{3} = 8$$

Beispiel 3:

$$\int_0^1\int_0^x\int_0^y x\,y^2 z\,dz\,dy\,dx = \int_0^1\int_0^x \frac{1}{2}xy^2z^2\Big|_0^y dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x x\,y^4\,dy\,dx = \frac{1}{10}\int_0^1 xy^5\Big|_0^x dx = \frac{1}{10}\int_0^1 x^6\,dx = \frac{1}{70}$$