Funktionen mehrerer Veränderlicher

Eine Funktion $f(x_1, \dots, x_n)$ heißt stetig bei $(x_1^0, \dots, x_n^0)$ falls

$$\lim_{(x_1,\dots,x_n) \to (x_1^0,\dots,x_n^0)} f(x_1, \dots, x_n) = f(x_1^0, \dots, x_n^0)$$

Beispiele:

i) $f(x,y) = x^2 + y^2$ stetig auf $\mathbb{R}^2$

ii) $f(x,y) = \cos(x \cdot y)$ "

iii) $$f(x,y) := \begin{cases} \dfrac{1-\cos(xy)}{y} & \text{für } y \neq 0 \\ 0 & y = 0 \end{cases}$$

Beweis durch Taylorentwicklung um $y = 0$

$$\cos(xy) = 1 - \frac{1}{2}(xy)^2 + \frac{1}{4!}(xy)^4 + \cdots$$

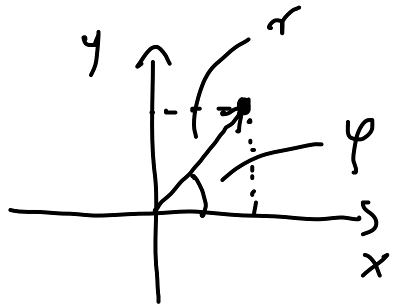
$$\Rightarrow \quad \frac{1-\cos(xy)}{y} = \frac{1}{2}x^2y - \frac{1}{4!}x^4y^3 + \cdots$$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{y} = 0 \qquad \text{auf allen Wegen !}$$

$\Rightarrow f(x,y)$ ist stetig

Manche Betrachtungen werden einfacher in geeignet gewählten Koordinatensystem

Beispiel: Polarkoordinaten



$$(x,y) \to (r,\varphi)$$

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned} \quad \Rightarrow \quad r = \sqrt{x^2+y^2} \; ; \; \varphi = \arctan\frac{y}{x}$$

Beispiel: i) $f(x,y) = \dfrac{1}{x^2+y^2} = \dfrac{1}{r^2}$

ii) $f(x,y) = xy\,\dfrac{x^2-y^2}{x^2+y^2} = r^2 \cos\varphi \sin\varphi \left(\cos^2\varphi - \sin^2\varphi\right)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} = \lim_{r\to 0} r^2 \cos\varphi \sin\varphi \left(\cos^2\varphi - \sin^2\varphi\right) = 0$$

Partielle Differentiation

Eine Funktion $f(x_1, .., x_n)$ heißt an der Stelle $(x_1^0, .., x_n^0)$ partiell nach x_i differenzierbar $1 \leq i \leq n$ falls die Funktion $g(x_i) := f(x_1^0, .., x_{i-1}^0, x_i, x_{i+1}^0, .., x_n^0)$ bei $x_i = x_i^0$ differenzierbar ist

"Die Variablen $x_1^0, .., x_{i-1}^0, x_{i+1}^0, .., x_n^0$ werden konstant gehalten"

$$\frac{dg}{dx_i} =: \frac{\partial f}{\partial x_i}(x_1^0, .., x_n^0) \qquad 1 \leq i \leq n$$

Beispiel i) $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x \; ; \; \frac{\partial f}{\partial y} = 2y$$

ii) $f(x,y) = \cos(xy)$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y) = -\sin(xy)\, y \; ; \; \frac{\partial f}{\partial y}(x,y) = -\sin(xy)\, x$$

Für partielle Ableitungen gelten Produktregel, Kettenregel, Quotientenregel

höhere partielle Ableitungen einer Funktion $f(x,y)$

Ableitungen 1. Ordnung $\frac{\partial f}{\partial x}(x,y)$; $\frac{\partial f}{\partial y}(x,y)$

Sind diese partiellen Ableitungen ihrerseits differenzierbar so gilt:

Ableitungen 2. Ordnung

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &:= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ \frac{\partial^2 f}{\partial y \partial x} &:= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \end{aligned}\right\} \text{ sind identisch falls sie stetig sind} \quad (\text{Schwarz})$$

Beispiel 1: $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x \quad ; \quad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad ; \quad \frac{\partial^2 f}{\partial x \, \partial y} = 0 \quad ; \quad \frac{\partial f}{\partial y \, \partial x} = 0$$

Beispiel 2: $f(x,y) = \cos(x \cdot y)$

Ableitungen 1. Ordnung: $\frac{\partial f}{\partial x} = -\sin(xy)\, y \quad ; \quad \frac{\partial f}{\partial y} = -\sin(xy)\, x$

〃 2. Ordnung: $\frac{\partial^2 f}{\partial x^2} = -\cos(xy)\, y^2 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = -\cos(xy)\, x^2$

$$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\sin(xy)\, x\right) = -\sin(xy) - \cos(xy)\, xy$$

$$\frac{\partial^2 f}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin(xy) - xy\cos(xy) = \frac{\partial^2 f}{\partial x \, \partial y}$$

Analog definiert man für $f(x_1, \dots, x_n)$

Ableitung 1. Ordnung $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}$

" 2. Ordnung: $\dfrac{\partial^2 f}{\partial x_1^2}, \dots, \dfrac{\partial^2 f}{\partial x_n^2}, \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ $\quad 1 \leq i, j \leq n$, $i \neq j$

Ableitung k-te Ordnung: $$\frac{\partial^k f}{\partial x_i^k} := \underbrace{\frac{\partial}{\partial x_i} \cdots \frac{\partial}{\partial x_i}}_{k\text{-mal}} f$$

Sowie alle gemischte Ableitungen

$n = 2$ $$\frac{\partial^k f}{\underbrace{\partial x \cdots \partial x}_{m} \underbrace{\partial y \cdots \partial y}_{n}} := \underbrace{\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x}}_{m\text{-mal}} \underbrace{\frac{\partial}{\partial y} \cdots \frac{\partial}{\partial y}}_{n\text{-mal}} f$$

$m + n = k$

Bsp. i) $f(x,y) = x^2 + y^2$

$$\frac{\partial^3 f}{\partial x^3} = 0 = \frac{\partial^3 f}{\partial y^3} \qquad \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = 0$$

ii) $f(x,y) = \cos(xy)$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(-\cos(xy)\, y^2\right) =$$

$$= \sin(xy)\, y^3$$

Taylorreihen

1 Veränderliche: $$f(x) = f(x_0) + (x-x_0)\frac{df}{dx}(x_0) + \frac{1}{2}(x-x_0)^2\frac{d^2f}{dx^2}(x_0) + \dots = \sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)$$

2 Veränderliche $f(x,y)$ sei beliebig oft partiell differenzierbar

Taylorentwicklung um (x_0, y_0)

$$f(x,y) = f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0)$$

$$+ \frac{1}{2}\Big[(x-x_0)^2\frac{\partial^2 f}{\partial x^2}(x_0,y_0) + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}(x_0,y_0) + $$

$$+ 2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x\,\partial y}(x_0,y_0)\Big] +$$

$$+ \frac{1}{3!}\Big[(x-x_0)^3\frac{\partial^3 f}{\partial x^3}(x_0,y_0) + (y-y_0)^3\frac{\partial^3 f}{\partial y^3}(x_0,y_0)$$

$$+ 3(x-x_0)^2(y-y_0)\frac{\partial^3 f}{\partial x^2 \partial y}(x_0,y_0) + 3(x-x_0)(y-y_0)^2\frac{\partial^3 f}{\partial y^2 \partial x}(x_0,y_0)\Big]$$

$+ \dots$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^k f(x_0,y_0)$$

Beispiel: $f(x,y) = \cos(x+y)$: Entwicklung um $(x_0,y_0) = (0,0)$

1\. Ordnung: $\frac{\partial f}{\partial x} = -\sin(x+y) = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial x}\Big|_{(0,0)} = \frac{\partial f}{\partial y}\Big|_{(0,0)} = 0$

2\. Ordnung: $\frac{\partial^2 f}{\partial x^2} = -\cos(x+y) = \frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial^2 f}{\partial x \partial y} = -\cos(x+y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = \frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} = \frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)} = -1$$

$$\Rightarrow \cos(x+y) = 1 - \frac{1}{2}\left(x^2 + y^2 + 2xy\right) + \dots$$

$$= 1 - \frac{1}{2}(x+y)^2 + \dots$$

Totales Differential

$f(x,y,z)$ Bewegung entlang einer Kurve $x(t), y(t), z(t)$

$\rightarrow f(x(t), y(t), z(t))$ hängt nur noch von t ab

$$\frac{df}{dt} \underset{\text{Kettenregel}}{=} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$ totale Ableitung

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$ $\rightarrow$ totales Differential

allgemein $f(x_1, \ldots, x_n)$

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} \qquad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Differentiation bei Koordinatenwechsel

Beispiel: $f(x,y) = x^2 + y^2$ $\qquad \frac{\partial f}{\partial x} = 2x \; ; \; \frac{\partial f}{\partial y} = 2y$

$f(r,\varphi) = r^2$ $\qquad \frac{\partial f}{\partial r} = 2r \; ; \; \frac{\partial f}{\partial \varphi} = 0$

$$f(x,y) = f\left(x(r,\varphi), y(r,\varphi)\right)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} =$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi}$$

$$x = r\cos\varphi \qquad \frac{\partial x}{\partial r} = \cos\varphi \qquad \frac{\partial y}{\partial r} = \sin\varphi$$

$$y = r\sin\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\cos\varphi$$

$$\Rightarrow \quad \frac{\partial f}{\partial r} = 2x\cos\varphi + 2y\sin\varphi = 2r \quad ✓$$

$$\frac{\partial f}{\partial \varphi} = 2x(-r\sin\varphi) + 2y\, r\cos\varphi = -2r\cos\varphi\sin\varphi + 2r\cos\varphi\sin\varphi$$

$$= 0 \quad ✓$$

Allgemein: Koordinatentransformation

$$(x_1, \dots, x_n) \to (y_1, \dots, y_n)$$

$$f(x_1, \dots, x_n) \to f(x_1(y_1, \dots, y_n), \dots, x_n(y_1, \dots, y_n))$$

$$\frac{\partial f}{\partial y_1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_n}$$

$$\boxed{\frac{\partial f}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y_j}}$$

Kettenregel bei Koordinatenwechsel

Gradient einer Funktion $f(x_1, .., x_n)$

Die ersten partiellen Ableitungen einer Funktion $f(x_1, .., x_n)$, also $\frac{\partial f}{\partial x_1}, .., \frac{\partial f}{\partial x_n}$ werden zu einem Vektor, dem Gradienten zusammengefasst:

$$\vec{\text{grad}}\, f(x_1, .., x_n) := \vec{e}_1 \frac{\partial f}{\partial x_1} + \vec{e}_2 \frac{\partial f}{\partial x_2} + .. \vec{e}_n \frac{\partial f}{\partial x_n} =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \underbrace{\vec{\nabla}}_{\text{Nabla-Operator}} f$$

$$\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \cdots \vec{e}_n \frac{\partial}{\partial x_n}$$

Beispiel: $f(x,y,z) = x^2 + y^2 + z^2$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = 2\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right)$$

$$= 2\vec{r}$$

Damit definiert man die <u>Richtungsableitung</u> einer Funktion in Richtung eines Einheitsvektors $\vec{u} = u_1 \vec{e}_1 + \cdots u_n \vec{e}_n$

$$\vec{u} \cdot \vec{\nabla} f = u_1 \frac{\partial f}{\partial x_1} + \cdots u_n \frac{\partial f}{\partial x_n} = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}$$

Beispiel: $\vec{\hat{r}} = \frac{1}{r}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right)$ radiale Einheitsvektor

$$\vec{\hat{r}} \cdot \vec{\nabla} f = \frac{2}{r}\left(x^2 + y^2 + z^2\right) = 2r$$