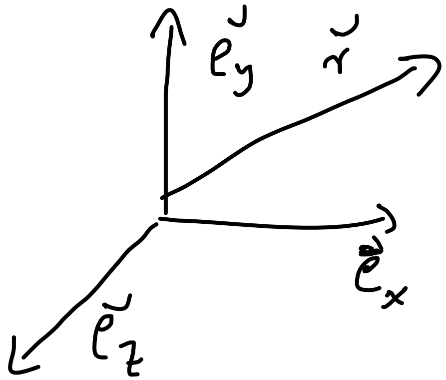
Vorlesung 12: Vektorwertige Funktionen

$f(t)$ $\quad\quad$ $\vec{r}(t)$

Massenpunkt in der Mechanik $\quad \vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$



$$\vec{r}(t) = x(t)\,\vec{e}_x + y(t)\,\vec{e}_y + z(t)\,\vec{e}_z$$

Basisvektoren $\vec{e}_x, \vec{e}_y, \vec{e}_z \;\hat{=}\; \vec{e}_1, \vec{e}_2, \vec{e}_3$ sind zeitunabhängig

Ableitung von $\underset{\sim}{r}(t)$:

$$\dot{\underset{\sim}{r}} := \frac{d\underset{\sim}{r}}{dt} = \dot{x}(t)\,\underset{\sim}{e}_x + \dot{y}(t)\,\underset{\sim}{e}_y + \dot{z}(t)\,\underset{\sim}{e}_z$$

$$= \frac{dx}{dt}\underset{\sim}{e}_x + \frac{dy}{dt}\underset{\sim}{e}_y + \frac{dz}{dt}\underset{\sim}{e}_z =$$

$$= \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}$$

vereinfachte Notation: $\dot{x} := \frac{dx}{dt}$; $\dot{y} := \frac{dy}{dt}$; $\dot{z} = \frac{dz}{dt}$

$\dot{\underset{\sim}{r}} = \underset{\sim}{v}$ = Geschwindigkeit des Massenpunkts

$\ddot{\underset{\sim}{r}} = \dot{\underset{\sim}{v}} = \underset{\sim}{a}$ = Beschleunigung

Rechenregeln:

i) $\frac{d}{dt}(\vec{a}+\vec{b}) = \dot{\vec{a}} + \dot{\vec{b}}$

ii) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \dot{\vec{a}}\cdot\vec{b} + \vec{a}\cdot\dot{\vec{b}}$

Beweis: $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}\Big(\underbrace{a_x b_x + a_y b_y + a_z b_z}_{=:\,\vec{a}\cdot\vec{b}}\Big) =$

$= \dot{a}_x b_x + a_x \dot{b}_x + \dot{a}_y b_y + a_y \dot{b}_y + \dot{a}_z b_z + a_z \dot{b}_z =$

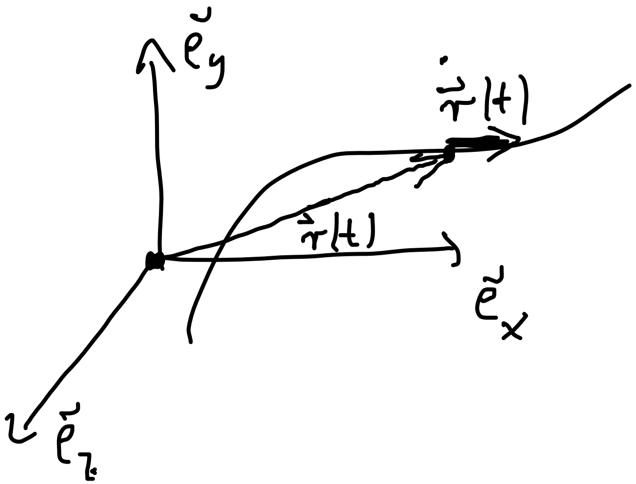
$= \dot{\vec{a}}\cdot\vec{b} + \vec{a}\cdot\dot{\vec{b}}$

da $\dot{\vec{a}}\cdot\vec{b} = \dot{a}_x b_x + \dot{a}_y b_y + \dot{a}_z b_z$

iii) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \dot{\vec{a}}\times\vec{b} + \vec{a}\times\dot{\vec{b}}$

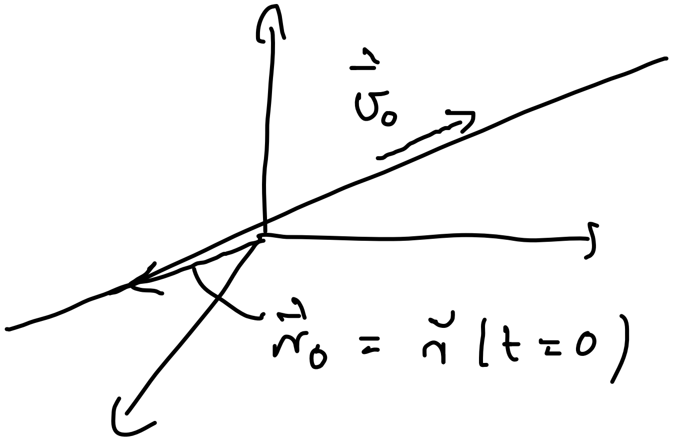
iv) $\frac{d}{dt}(\alpha\,\vec{a}) = \frac{d\alpha}{dt}\,\vec{a} + \alpha\,\dot{\vec{a}}$

Bahn eines Teilchens $\rightarrow$ Raumkurve



$\vec{v}(t) = \dot{\vec{r}}(t)$ = Tangentenvektor am Punkt $\vec{r}(t)$

Beispiel: $\quad \vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t \quad \Rightarrow \quad \vec{v}(t) = \dot{\vec{r}}(t) = \vec{v}_0$



$$\vec{r}(t) = \vec{r}_0 + \left(v_{0x}\vec{e}_x + v_{0y}\vec{e}_y + v_{0z}\vec{e}_z\right)t$$

$$\Rightarrow \dot{\vec{r}}(t) = v_{0x}\vec{e}_x + v_{0y}\vec{e}_y + v_{0z}\vec{e}_z = \vec{v}_0$$

$$\ddot{\vec{r}}(t) = \vec{0} = \vec{a}$$

Das beschreibt den Fall einer kraftfreien Bewegung

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$ → Für $\vec{v}_0$ beliebig aber fixiert ist dies die allgemeine Lösung der Newtonschen Gleichung für $\vec{F} = \vec{0}$

$$m\vec{a} = \vec{0} \iff m\ddot{\vec{r}} = 0 \iff \ddot{\vec{r}} = 0$$

Beispiel 2: konstante Kraft

$$\vec{F} = q\vec{E}_0 \qquad \vec{E}_0 = E_0 \vec{e}_z \qquad \dot{\vec{E}}_0 = 0$$

homogenes elektrisches Feld zeitunabhängig

Newtonsche Gleichung

$$m\ddot{\vec{r}} = \vec{F} = q\vec{E}_0 \Rightarrow \ddot{\vec{r}} = \frac{q\vec{E}_0}{m} = \text{const.}$$

$$\dot{\vec{r}} = \frac{q\vec{E}_0}{m} t + \vec{v}_0 \qquad \vec{v}_0 = \text{const.}$$

$$\vec{r}(t) = \frac{1}{2}\,\frac{q\vec{E}_0}{m}\,t^2 + \vec{v}_0 t + \vec{r}_0$$

zwei Anfangsbedingungen

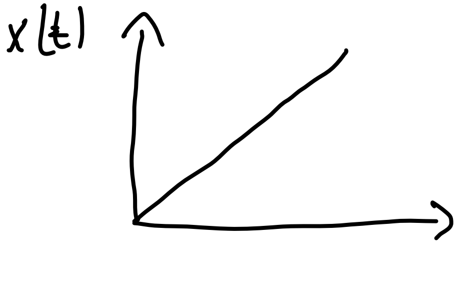
Probe: $\dot{\vec{r}} = \frac{q\vec{E}_0}{m}\,t + \vec{v}_0$

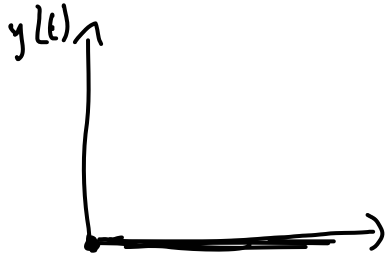
in Komponenten: $\vec{E}_0 = E_0\,\vec{e}_z$; $\vec{r}_0 = \vec{0}$; $\vec{v}_0 = v_{0x}\,\vec{e}_x$

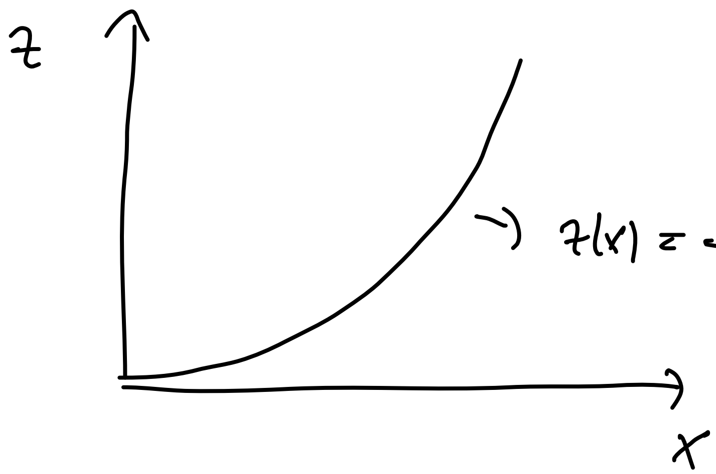
$$x(t) = v_{0x} t + r_{0x} = v_{0x} t \Rightarrow t = \frac{x}{v_{0x}}$$

$$y(t) = 0$$

$$z(t) = \frac{1}{2}\frac{qE_0}{m} t^2 = \frac{qE_0}{2m}\left(\frac{x}{v_{0x}}\right)^2$$

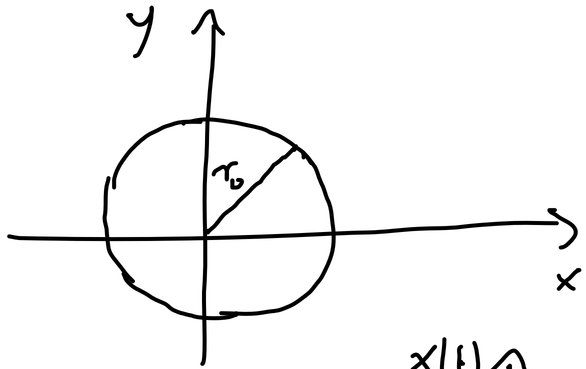






$\rightarrow \quad z(x) = \frac{qE_0}{2m}\left(\frac{x}{v_{0,x}}\right)^2$

Beispiel 2: Kreisbewegung

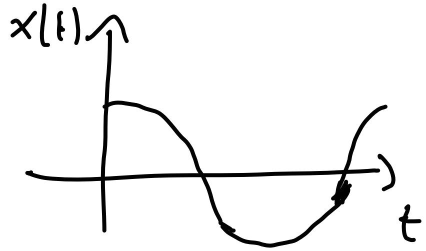


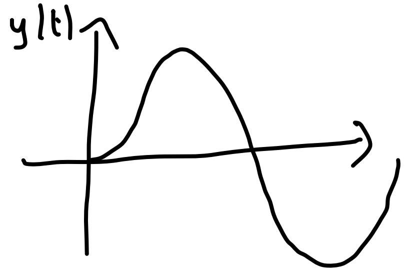
$$x(t) = r_0 \cos \omega t$$

$$y(t) = r_0 \sin \omega t$$

$$r_0 = \text{const.}$$

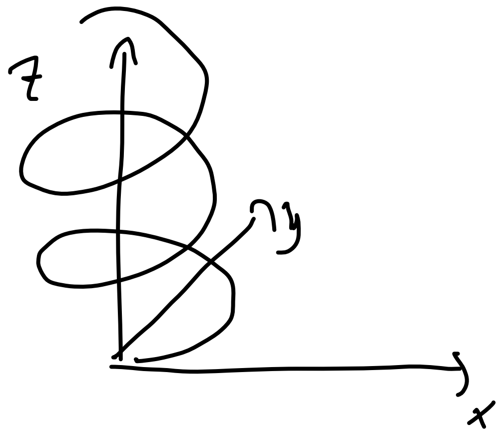
$$x^2(t) + y^2(t) = r_0^2$$





$$\vec{r}(t) = r_0 \left(\cos \omega t \cdot \vec{e}_x + \sin \omega t \, \vec{e}_y \right)$$

Beispiel 3 :



$$x(t) = r_0 \cos \omega t$$
$$y(t) = r_0 \sin \omega t$$
$$z(t) = v_0 \, t$$

Schraubenlinie beschreibt die Bewegung eines geladenen Teilchens in einem konstanten Magnetfeld $\vec{B}_0$

Ladung q und Magnetfeld $\vec{B}$ Lorentzkraft

$$\vec{F} = q \, \vec{v} \times \vec{B} = q \, \dot{\vec{r}} \times \vec{B}$$

Behauptung: Schraubenbewegung ist (allgemeine) Lösung der Newtonschen Gleichung für eine Lorentzkraft

$$m\ddot{\vec{r}} = q(\dot{\vec{r}} \times \vec{B}) \qquad \vec{B} = -B_0\vec{e}_z$$

$$\dot{x}(t) = -r_0\omega\sin\omega t \quad ; \quad \dot{y}(t) = r_0\omega\cos\omega t$$

$$\ddot{x}(t) = -r_0\omega^2\cos\omega t = -\omega^2 x(t) = -\omega\dot{y}$$

$$\ddot{y}(t) = -r_0\omega^2\sin\omega t = -\omega^2 y(t) = \omega\dot{x}$$

$$\dot{z}(t) = v_0 \quad ; \quad \ddot{z}(t) = 0$$

$$\vec{F}_{Lorentz} = q(\dot{\vec{r}} \times \vec{B}) = -qB_0(\dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z) \times \vec{e}_z$$

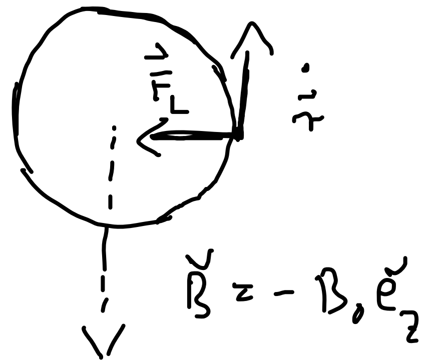
$$= -qB_0(-\dot{x}\vec{e}_y + \dot{y}\vec{e}_x)$$

$$m\ddot{\underset{\sim}{r}} = m(\ddot{x}\underset{\sim}{e}_x + \ddot{y}\underset{\sim}{e}_y + \ddot{z}\underset{\sim}{e}_z) = m\omega(\dot{x}\underset{\sim}{e}_y - \dot{y}\underset{\sim}{e}_x)$$

$$\Rightarrow \quad m\omega(\dot{x}\underset{\sim}{e}_y - \dot{y}\underset{\sim}{e}_x) = qB_0(\dot{x}\underset{\sim}{e}_y - \dot{y}\underset{\sim}{e}_x)$$

$$\Rightarrow \quad \omega = \frac{qB_0}{m}$$ $\rightarrow$ Larmor-Frequenz

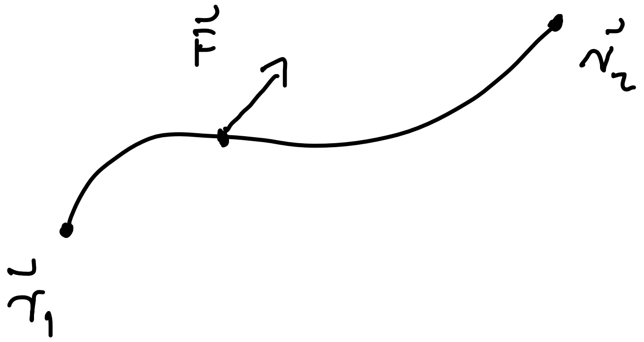
$q > 0$



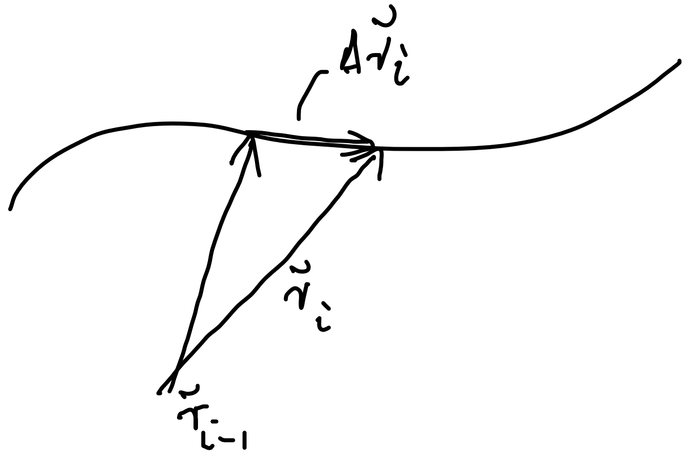
Kurvenintegral:

Motivation: $\vec{r}(t)$ Kraft $\vec{F}$ $\qquad d\vec{r} = dx\,\vec{e}_x + dy\,\vec{e}_y + dz\,\vec{e}_z$

Arbeit: $$W := -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r} := -\int_{t_1}^{t_2} \left[\vec{F}(\vec{r}(t)) \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\dot{\vec{r}} = \vec{v}(t)} \right] dt$$



$$\int \vec{F}(\vec{r}) \cdot d\vec{r} = \lim_{N\to\infty} \sum_{i=1}^{N} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i = \lim_{N\to\infty} \sum_{i=1}^{N} \vec{F}(\vec{r}_i) \frac{\Delta\vec{r}_i}{\Delta t_i} \Delta t_i$$

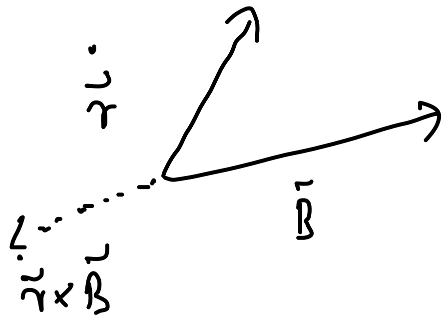


Grenzübergang: $\lim_{N\to\infty} \Delta t_i = \lim_{N\to\infty} \Delta \tilde{r}_i = 0$

Beispiel: Arbeit verrichtet durch die Lorentzkraft

$$W = -\int_{t_1}^{t_2} \tilde{F}_{\text{Lorentz}}(\tilde{r}(t)) \cdot \dot{\tilde{r}}(t)\, dt$$

$$\tilde{F}_{\text{Lorentz}} = q(\dot{\tilde{r}} \times \tilde{B})$$



$$\dot{\vec r} \perp (\dot{\vec r} \times \vec B) \quad \Rightarrow \quad \dot{\vec r} \cdot (\dot{\vec r} \times \vec B) \equiv 0$$

Beispiel 1: Teilchen im homogenen elektrischen Feld

$$\vec F = q \vec E_0$$

$$W = - \int_{\vec r_1}^{\vec r_2} \vec F \cdot d\vec r = - \int_{t_1}^{t_2} \vec F \cdot \frac{d\vec r}{dt}\, dt = -q \int \vec E_0 \cdot \dot{\vec r}\, dt$$

$$= -q \int_{t_1}^{t_2} \vec E_0 \cdot \left(\frac{q}{m} \vec E_0 t + \vec v_0 \right) dt = - \frac{q^2}{2m} \left| \vec E_0 \right|^2 \left(t_2^2 - t_1^2 \right) - q \vec E_0 \cdot \vec v_0 (t_2 - t_1)$$