Rotation:

$$\breve{\nabla} \times \breve{A} = \sum_i \frac{\breve{u}_i}{b_i} \frac{\partial}{\partial y_i} \times \sum_j A_j \breve{u}_j$$

$$= \sum_{i,j} \frac{1}{b_i} \frac{\partial A_j}{\partial y_i} \breve{u}_i \times \breve{u}_j + \sum_{i,j} \frac{A_j}{b_i} \breve{u}_i \times \frac{\partial \breve{u}_j}{\partial y_i}$$

Zwischenrechnung: Betrachte $\breve{u}_i \times \frac{\partial \breve{u}_j}{\partial y_i}$ für $j=1$

$$\breve{u}_1 = \breve{u}_2 \times \breve{u}_3 \Rightarrow \boxed{\frac{\partial \breve{u}_1}{\partial y_i} = \frac{\partial \breve{u}_2}{\partial y_i} \times \breve{u}_3 + \breve{u}_2 \times \frac{\partial \breve{u}_3}{\partial y_i}}$$

$i=1$:

$$\breve{u}_1 \times \frac{\partial \breve{u}_1}{\partial y_1} = \frac{\partial \breve{u}_2}{\partial y_1} \underbrace{(\breve{u}_1 \cdot \breve{u}_3)}_{=0} - \breve{u}_3 \left(\breve{u}_1 \cdot \frac{\partial \breve{u}_2}{\partial y_1}\right) + \breve{u}_2 \left(\breve{u}_1 \cdot \frac{\partial \breve{u}_3}{\partial y_1}\right) - \frac{\partial \breve{u}_3}{\partial y_1} \underbrace{(\breve{u}_1 \cdot \breve{u}_2)}_{=0}$$

bac-cab Regel

$$= -\breve{u}_3 \left(\frac{\partial \breve{u}_2}{\partial y_1} \cdot \breve{u}_1\right) + \breve{u}_2 \left(\frac{\partial \breve{u}_3}{\partial y_1} \cdot \breve{u}_1\right) = (**)$$

$$= -\breve{u}_3 \left(\frac{1}{b_2} \frac{\partial b_1}{\partial y_2}\right) + \breve{u}_2 \left(\frac{1}{b_3} \frac{\partial b_1}{\partial y_3}\right)$$

$i=2$:

$$\breve{u}_2 \times \frac{\partial \breve{u}_1}{\partial y_2} = -\breve{u}_3 \underbrace{\left(\frac{\partial \breve{u}_2}{\partial y_2} \cdot \breve{u}_2\right)}_{= \frac{1}{2} \frac{\partial}{\partial y_2} (\breve{u}_2^2) = 0} + \breve{u}_2 \left(\frac{\partial \breve{u}_3}{\partial y_2} \cdot \breve{u}_2\right) - \frac{\partial \breve{u}_3}{\partial y_2}$$

$$= -\breve{u}_1 \left(\frac{\partial \breve{u}_3}{\partial y_2} \cdot \breve{u}_1\right) - \breve{u}_3 \underbrace{\left(\frac{\partial \breve{u}_3}{\partial y_2} \cdot \breve{u}_3\right)}_{= \frac{1}{2} \frac{\partial}{\partial y_2} (\breve{u}_3^2) = 0} = -\breve{u}_1 \left(\frac{\partial \breve{u}_3}{\partial y_2} \cdot \breve{u}_1\right)$$

$i=3$:

$$\breve{u}_3 \times \frac{\partial \breve{u}_1}{\partial y_3} = \frac{\partial \breve{u}_2}{\partial y_3} - \breve{u}_3 \left(\frac{\partial \breve{u}_2}{\partial y_3} \cdot \breve{u}_3\right) = \breve{u}_1 \left(\frac{\partial \breve{u}_2}{\partial y_3} \cdot \breve{u}_1\right) + \breve{u}_2 \underbrace{\left(\frac{\partial \breve{u}_2}{\partial y_3} \cdot \breve{u}_2\right)}_{=0}$$

$$= \breve{u}_1 \left(\frac{\partial \breve{u}_2}{\partial y_3} \cdot \breve{u}_1\right)$$

$$\Rightarrow j=1 \quad \sum_i \frac{1}{b_i}\breve{u}_i \times \frac{\partial \breve{u}_1}{\partial y_i} = -\frac{1}{b_1 b_2}\frac{\partial b_1}{\partial y_2}\breve{u}_3 + \frac{1}{b_1 b_3}\frac{\partial b_1}{\partial y_3}\breve{u}_2$$

$$+\left[-\frac{1}{b_2}\left(\frac{\partial \breve{u}_3}{\partial y_2}\cdot\breve{u}_1\right)+\frac{1}{b_3}\left(\frac{\partial \breve{u}_2}{\partial y_3}\cdot\breve{u}_1\right)\right]\breve{u}_1$$

aber aus (*) Folgt mit $i=3,\ j=2$

$$b_3\frac{\partial \breve{u}_3}{\partial y_2}+\frac{\partial b_3}{\partial y_2}\breve{u}_3 = b_2\frac{\partial \breve{u}_2}{\partial y_3}+\frac{\partial b_2}{\partial y_3}\breve{u}_2$$

multipliziere mit $\frac{\breve{u}_1}{b_2 b_3}$ $\Rightarrow$ $\frac{1}{b_3}\left(\frac{\partial \breve{u}_2}{\partial y_3}\cdot\breve{u}_1\right)-\frac{1}{b_2}\left(\frac{\partial \breve{u}_3}{\partial y_2}\cdot\breve{u}_1\right)=0$

also

$$\sum_i \frac{1}{b_i}\breve{u}_i \times \frac{\partial \breve{u}_1}{\partial y_i} = -\frac{1}{b_1 b_2}\frac{\partial b_1}{\partial y_2}\breve{u}_3 + \frac{1}{b_1 b_3}\frac{\partial b_1}{\partial y_3}\breve{u}_2$$

und damit ~~insgesamt:~~ durch zyklische Vertauschung allgemein:

$$\sum_i \frac{1}{b_i}\breve{u}_i \times \frac{\partial \breve{u}_j}{\partial y_i} = \frac{1}{b_1 b_2 b_3}\varepsilon_{ijk}\frac{\partial b_j}{\partial y_i} b_k \breve{u}_k$$

$$\cancel{\Rightarrow \breve{\nabla}\times\breve{A} = \frac{1}{b_1 b_2 b_3}\Big[}$$

$$\breve{\nabla}\times\breve{A} = \sum_{i,j,k}\frac{\varepsilon_{ijk}}{b_1 b_2 b_3}\left(b_k b_j \frac{\partial A_j}{\partial y_i}\breve{u}_k + \frac{\partial b_j}{\partial y_i}A_j b_k \breve{u}_k\right)$$

$$= \frac{1}{b_1 b_2 b_3}\sum_{i,j,k}\frac{\partial (b_j A_j)}{\partial y_i} b_k \breve{u}_k\, \varepsilon_{ijk}$$

oder $\left(\breve{\nabla}\times\breve{A}\right)_k = \frac{b_k}{b_1 b_2 b_3}\sum_{i,j}\varepsilon_{ijk}\frac{\partial (b_j A_j)}{\partial y_i}$

Bemerkung:

da die $\frac{\partial \vec{r}}{\partial y_i}$ paarweise orthogonal sind

$$b_1 b_2 b_3 = \left|\frac{\partial \vec{r}}{\partial y_1}\right| \left|\frac{\partial \vec{r}}{\partial y_2}\right| \left|\frac{\partial \vec{r}}{\partial y_3}\right| = \left| \left(\frac{\partial \vec{r}}{\partial y_1} \times \frac{\partial \vec{r}}{\partial y_2}\right) \cdot \frac{\partial \vec{r}}{\partial y_3} \right|$$

$$= \left| \det \begin{pmatrix} \frac{\partial x}{\partial y_1} & \frac{\partial y}{\partial y_1} & \frac{\partial z}{\partial y_1} \\ \frac{\partial x}{\partial y_2} & \frac{\partial y}{\partial y_2} & \frac{\partial z}{\partial y_2} \\ \frac{\partial x}{\partial y_3} & \frac{\partial y}{\partial y_3} & \frac{\partial z}{\partial y_3} \end{pmatrix} \right| = \left| \frac{\partial(x, y, z)}{\partial(y_1, y_2, y_3)} \right|$$

= Funktionaldeterminante der Transformation $\vec{r}(y_1, y_2, y_3)$

Laplace-Operator:

$$\Delta = \vec{\nabla}^2 = \frac{1}{b_1 b_2 b_3} \left[\sum_i \frac{\partial}{\partial y_i} \left(\frac{b_1 b_2 b_3}{b_i^2} \frac{\partial}{\partial y_i} \right) \right]$$

$$= \frac{1}{b_1 b_2 b_3} \left[\frac{\partial}{\partial y_1} \left(\frac{b_2 b_3}{b_1} \frac{\partial}{\partial y_1} \right) + \frac{\partial}{\partial y_2} \left(\frac{b_1 b_3}{b_2} \frac{\partial}{\partial y_2} \right) + \frac{\partial}{\partial y_3} \left(\frac{b_1 b_2}{b_3} \frac{\partial}{\partial y_3} \right) \right]$$

Beispiel 1: Sphärische Polarkoordinaten (r, θ, φ)

$$\vec{r}(r, \theta, \varphi) = r(\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$b_r = 1 \; ; \quad b_\theta = r \; ; \quad b_\varphi = r \sin\theta$$

$$\Rightarrow \quad \vec{\nabla} \phi = \vec{u}_r \frac{\partial \phi}{\partial r} + \vec{u}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \vec{u}_\varphi \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \varphi}$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = \frac{\vec{u}_r}{b_\theta b_\varphi} \left[\frac{\partial(b_\varphi A_\varphi)}{\partial \theta} - \frac{\partial(b_\theta A_\theta)}{\partial \varphi} \right]$$

$$+ \frac{\vec{u}_\theta}{b_r b_\varphi} \left[\frac{\partial(b_r A_r)}{\partial \varphi} - \frac{\partial(b_\varphi A_\varphi)}{\partial r} \right]$$

$$+ \frac{\vec{u}_\varphi}{b_r b_\theta} \left[\frac{\partial(b_\theta A_\theta)}{\partial r} - \frac{\partial(b_r A_r)}{\partial \theta} \right] =$$

$$= \frac{\breve{u}_r}{r^2 \sin\theta}\left[\frac{\partial(r\sin\theta\, A_\varphi)}{\partial\theta} - \frac{\partial(r\, A_\theta)}{\partial\varphi}\right]$$

$$+ \frac{\breve{u}_\theta}{r\sin\theta}\left[\frac{\partial A_r}{\partial\varphi} - \frac{\partial(r\sin\theta\, A_\varphi)}{\partial r}\right]$$

$$+ \frac{\breve{u}_\varphi}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

$$= \frac{\breve{u}_r}{r\sin\theta}\left[\frac{\partial(\sin\theta\, A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right] + \frac{\breve{u}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(r A_\varphi)}{\partial r}\right]$$

$$+ \frac{\breve{u}_\varphi}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

Beispiel 2: Zylinderkoordinaten ρ, φ, z

$$\breve{r}(\rho,\varphi,z) = (\rho\cos\varphi, \rho\sin\varphi, z)$$

$$b_\rho = 1, \quad b_\varphi = \rho, \quad b_z = 1$$

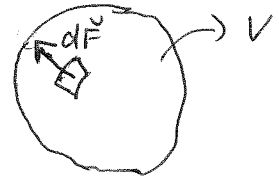
$$\Rightarrow \breve{\nabla}\phi = \breve{u}_\rho \frac{\partial\phi}{\partial\rho} + \frac{\breve{u}_\varphi}{\rho}\frac{\partial\phi}{\partial\varphi} + \breve{u}_z\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \breve{\nabla}\times\breve{A} = \breve{u}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\rho}{\partial z}\right) + \breve{u}_\varphi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \frac{\breve{u}_z}{\rho}\left[\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right]$$

2.6. Integralform der Maxwell'schen Gleichungen

Gauß'sches Gesetz $\quad \vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\varepsilon_0}$

$$\Rightarrow \quad \underbrace{\frac{1}{\varepsilon_0}\int_V \rho_e \, dV}_{= Q} = \int_V \vec{\nabla}\cdot\vec{E} = \int_{\partial V} \vec{E}\cdot d\vec{F}$$



$$\Rightarrow \quad \frac{\text{Ladung in } V}{\varepsilon_0} = \int_{\partial V} \vec{E}\cdot d\vec{F} = \text{Fluß von } \vec{E} \text{ durch Randfläche von } V$$

Induktionsgesetz $\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \quad -\frac{\partial}{\partial t}\int_F \vec{B}\cdot d\vec{F} = \int_F (\vec{\nabla}\times\vec{E})\cdot d\vec{F} = \oint_{\partial F} \vec{E}\cdot d\vec{r}$$

$$\Rightarrow \quad -\frac{\partial}{\partial t}\left(\text{Fluß von } \vec{B} \text{ durch } F\right) = \text{Umlaufintegral von } \vec{E} \text{ um Rand von } F$$

Gauß'sches Gesetz des Magnetismus $\quad \vec{\nabla}\cdot\vec{B} = 0$

$$\Rightarrow \quad \int_{\partial V} \vec{B}\cdot d\vec{F} = 0 \; ; \quad \text{Fluß von } \vec{B} \text{ durch geschlossene Fläche} = 0$$

Ampèresches Gesetz $\quad c_0^2\, \vec{\nabla}\times\vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \quad \frac{1}{\varepsilon_0}\int_F \vec{j}\cdot d\vec{F} + \frac{\partial}{\partial t}\int_F \vec{E}\cdot d\vec{F} = c_0^2 \int_F \vec{\nabla}\times\vec{B} = c_0^2 \oint_{\partial F} \vec{B}\cdot d\vec{r}$$

$$\Rightarrow \quad c_0^2\left(\text{Umlaufintegral von } \vec{B} \text{ um Rand von } F\right) =$$

$$= \frac{\text{Strom durch } F}{\varepsilon_0} + \frac{\partial}{\partial t}\Big(\underbrace{\text{Fluß von } \vec{E} \text{ durch } F}_{\hat{=}\ \text{„Verschiebungsstrom"}}\Big)$$

Kontinuitätsgleichung: $\frac{\partial \rho_e}{\partial t} + \breve{\nabla}\cdot\breve{j} = 0$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial t}\int_V \rho_e \, dV}_{= Q} = -\int_V \breve{\nabla}\cdot\breve{j} = -\int_{\partial V} \breve{j}\cdot d\breve{F}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\text{Ladung in } V\right) = -\left(\text{Strom aus } V \text{ heraus}\right)$$

2.7. Nochmal Wegunabhängigkeit von Kurvenintegralen

$$\left.\begin{aligned} &\int_C \breve{A}\cdot d\breve{r} \text{ wegunabhängig bei fixen Randpunkten} \\ \Leftrightarrow \quad &\oint \breve{A}\cdot d\breve{r} = 0 \quad \forall \text{ Schleifen} \\ \Leftrightarrow \quad &\breve{\nabla}\times\breve{A} = 0 \end{aligned}\right\} \text{ für "einfach zusammenhängende" Gebiete}$$

erste Äquivalenz bereits früher gezeigt

zweite Äquivalenz:

$$\Rightarrow: \quad \breve{n}\cdot(\breve{\nabla}\times\breve{A}) = \lim_{\Delta F\to 0}\frac{1}{\Delta F}\oint_{\partial(\Delta F)} \breve{A}\cdot d\breve{r} = 0$$

$$\Leftarrow: \quad 0 = \int (\breve{\nabla}\times\breve{A})\cdot d\breve{F} = \oint_{\partial F} \breve{A}\cdot d\breve{r} = 0$$

gleichbedeutend mit Aussage daß $\breve{A}\cdot d\breve{r} = A_x dx + A_y dy + A_z dz$ totales Differential ist

Anmerkung: Zerlegungssatz (ohne Beweis):

Ein beliebiges Vektorfeld $\breve{A}$ kann zerlegt werden in

$$\breve{A} = \breve{A}_{div} + \breve{A}_{rot}$$

mit

$$\operatorname{div}\breve{A} = \operatorname{div}\breve{A}_{div}; \quad \operatorname{rot}\breve{A} = \operatorname{rot}\breve{A}_{rot}, \quad \text{also}$$

$$\operatorname{div}\breve{A}_{rot} = 0 = \operatorname{rot}\breve{A}_{div}$$

3. Elektrostatik, Magnetostatik und die Poisson Gleichung

3.1. Rolle der Poissongleichung

Elektrostatik: $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$

$\Rightarrow$ $\vec{E}$ kann als $\vec{E} = -\vec{\nabla}\phi$ geschrieben werden

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \quad \Rightarrow \quad \Delta \phi = -\frac{\rho_e}{\varepsilon_0}$$

Magnetostatik: $\quad \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ $\vec{B}$ kann geschrieben werden

als $\quad \vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{A}$ = Vektorpotential

$$c_0^2 \, \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A}$$

$$\Rightarrow \quad \Delta \vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = -\frac{\vec{j}}{c_0^2 \varepsilon_0} = -\frac{\vec{j}}{\mu_0} \qquad (*)$$

magnetische Suszeptibilität

$\vec{A}$ ist nicht eindeutig bestimmt, da man eine "Eichtransformation"

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\varphi = \vec{A}'$$

ausführen kann, ohne $\vec{B} = \vec{\nabla} \times \vec{A}$ zu verändern. Man kann $\vec{A}'$ sogar so wählen daß $\vec{\nabla} \cdot \vec{A}' = 0$:

$$0 = \vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \Delta\varphi \Rightarrow$$

Man muß dazu eine Lösung von $\Delta\varphi = -\vec{\nabla} \cdot \vec{A}$ finden

Damit folgt:

$$\Delta \vec{A} = -\frac{\vec{j}}{\mu_0}$$

In obigen Überlegungen trat die Poissongleichung 3 mal auf $\Rightarrow$ Wir benötigen nun Lösungsmethoden