$$G(\bar{r},\bar{r}') = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} Y^*_{lm}(\theta',\varphi')(r')^l \qquad (r > r').$$

Faktor: $\frac{4\pi}{2l+1}$:

$$g_l(r,r') = C\left[r_<^{\,l} + \frac{1}{r_>^{\,l+1}} \right]$$

$$\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - l(l+1) \right] g_l(r,r') = 4\pi\, \delta(r-r')$$

$$\int_{r'-\varepsilon}^{r'+\varepsilon} dr \left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - l(l+1) \right] g_l(r,r') = 4\pi.$$

Einsetzen: $g_l(r,r') = C\left(r_<^{\,l} + \frac{1}{r_>^{\,l+1}} \right) \rightarrow$

$$\rightarrow C = \frac{4\pi}{2l+1}.$$

Potential:

$$\Phi(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \, q_{lm}$$

$$q_{lm} = \int_{V'} \rho(r', \theta', \varphi')\,(r')^{l}\, Y_{lm}^{*}(\theta', \varphi') \cdot r'^2 \, dr' \sin\theta' \, d\theta' \, d\varphi'$$

q_{lm} – Multipolmoment.

$l = 0 \longrightarrow$ Monopol : $q_{00} = \int_{V'} \rho(\vec{r}\,')\, d^3r' = Q.$

$l = 1$: q_{11}, q_{10}, $q_{1,-1}$ $\Longleftrightarrow$ p_x, p_y, p_z – Dipolmoment

$l = 2$: $m = -2, -1, 0, 1, 2$ – Quadrupol

$$l=0: \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l=1: \quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\varphi} \\ Y_{10} = \sqrt{\frac{3}{8\pi}} \cos\theta \end{cases}$$

$$\left[Y_{l,-m} = (-1)^m Y_{lm}^* \right] \longmapsto Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

$$l=2: \quad \begin{cases} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\varphi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\varphi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) \end{cases}$$

Bezug zu kartesischen Komponenten:

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$

$$r\,Y_{11}^* = -\sqrt{\frac{3}{8\pi}}\, r\sin\theta\, e^{-i\varphi} =$$
$$= -\sqrt{\frac{3}{8\pi}}\, r\sin\theta(\cos\varphi - i\sin\varphi) =$$
$$= -\sqrt{\frac{3}{8\pi}}\,(x - iy).$$

Multipolentwicklung in Magnetostatik.

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$0 = \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{j} \quad \Rightarrow \quad \nabla \cdot \vec{j} = 0.$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}$$

$\vec{A}(\vec{r})$ – Vektorpotential.

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{j}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Eichinvarianz: $\vec{A}(\vec{r}) \longmapsto \vec{A}(\vec{r}) + \nabla \chi(\vec{r}) \quad \Rightarrow \quad \vec{B}(\vec{r})$ unverändert.

$$\vec{B} = \nabla \times (\vec{A} + \nabla \chi) = \nabla \times \vec{A} + \underbrace{\nabla \times (\nabla \chi)}_{= 0} = \nabla \times \vec{A}.$$

Fixiere $\chi(\vec{r})$ durch: $\nabla \cdot \vec{A} = 0$ – Coulomb-Eichung.

$$-\nabla^2 \vec{A} = \mu_0 \vec{j} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{j}$$

in kartesischen Koordinaten: $\nabla^2 A_i = -\mu_0 j_i \quad (i = x, y, z)$.

Lösung: $$A_i(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{j_i(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r'$$

Multipolentwicklung:

$$A_i(\vec{r}) = \mu_0 \sum_{l=0}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \times$$

$$\times \int_{V'} j_i(r', \theta', \varphi') (r')^l \, Y^*_{lm}(\theta', \varphi') \, dr' \sin\theta' d\theta' d\varphi' \, r'^2$$