Greensfunktion

$$\nabla^2_{r_2} G(\bar r, \bar r') = -4\pi\,\delta(\bar r - \bar r') \qquad \Big|\; G(\bar r, \bar r') \xrightarrow[\substack{\bar r \to \infty \\ \bar r' \to \infty}]{} 0$$

vgl:
$$\underline{\nabla^2 \phi(\bar r) = -\frac{1}{\varepsilon_0}\,\delta(\bar r)} \quad \text{– Punktladung}$$

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\bar r|}$$

$$4\pi\varepsilon_0 \cdot \nabla^2 \phi(\bar r - \bar r') = -4\pi\,\delta(\bar r - \bar r') \qquad \phi(\bar r - \bar r') \xrightarrow[|\bar r - \bar r'| \to \infty]{} 0$$

$$G(\bar r, \bar r') = \frac{1}{|\bar r - \bar r'|}$$

$$\delta(\bar{r}-\bar{r}') = \frac{1}{r^2}\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta').$$

$$\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} G(\bar{r},\bar{r}') =$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

Ansatz:

$$G(\bar{r},\bar{r}') = \sum_{l=0}^{\infty} g_l(r,r')\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{lm}(\theta,\varphi) = -l(l+1)\,Y_{lm}(\theta,\varphi)$$

$$\nabla_r^2 G(\vec{r},\vec{r}\,') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) \times$$

$$\times \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2} \right] g_l(r,r') =$$

$$= \frac{1}{r^2}\,\delta(r-r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi). \qquad (*)$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, Y^*_{lm}(\theta,\varphi) \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta'\, d\theta'\, Y_{lm}(\theta',\varphi') \times \quad \Big| \quad \ldots - \text{''} - \quad (*)$$

$$\boxed{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, Y^*_{lm}(\theta,\varphi) \cdot Y_{l'm'}(\theta,\varphi) = \#\, \delta_{ll'}\, \delta_{mm'}}$$

$\forall l$:

$$\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - l(l+1) \right] g_l(r,r') = \delta(r-r')$$

Vgl: $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \ell(\ell+1)\, V(r)$$

Lösung: $$V(r) = A\, r^{\ell} + B\, r^{-\ell-1}.$$

$$\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \ell(\ell+1)\right] g_\ell(r,r') = \delta(r-r').$$

$r \neq r'$: $$g_\ell(r,r') = A_\ell(r')\, r^{\ell} + B_\ell(r')\, r^{-\ell-1}$$

Symmetrie $r \leftrightarrow r'$: $$g_\ell(r,r') = A'_\ell(r)\, r'^{\ell} + B'_\ell(r)(r')^{-\ell-1}.$$

$$A_\ell(r')\, r^{\ell} = B'_\ell(r)\,(r')^{-\ell-1} \implies \begin{array}{l} A_\ell(r') = c_\ell \cdot (r')^{-\ell-1} \\ B'_\ell(r) = c_\ell \cdot r^{\ell} \end{array}$$

$$\left|\begin{array}{l} A'_\ell(r)\, r'^{\ell} = B_\ell(r')\, r^{-\ell-1} \\ A'_\ell(r) = d_\ell\, r^{-\ell-1} \\ B_\ell(r') = d_\ell (r')^{\ell}. \end{array}\right.$$

$$g_\ell(r,r') = c_1 \frac{r^{\ell}}{(r')^{\ell+1}} + c_2 \frac{(r')^{\ell}}{r^{\ell+1}}$$

$$g_\ell(r, r') = C_1 \frac{r^\ell}{(r')^{\ell+1}} + C_2 \frac{(r')^\ell}{r^{\ell+1}} .$$

1) $r \to \infty$: r' - fixiert. $g_\ell(r, r') \to 0$

$$\underline{C_1 = 0, \quad r > r'}$$

2) $r' \to \infty$, r - fixiert $(r' > r)$: $g_\ell(r, r') \mapsto 0$.

$$C_2 = 0, \quad r' > r$$

$$g_\ell(r, r') = \begin{cases} C_2 \dfrac{(r')^\ell}{r^{\ell+1}}, & r > r' \\ C_1 \dfrac{r^\ell}{(r')^{\ell+1}}, & r < r'. \end{cases}$$

$$G(\bar r - \bar r') = \frac{1}{|\bar r - \bar r'|} .$$

Betrachte $\bar r \parallel \bar r'$

$$G(\bar r - \bar r') = \frac{1}{|r - r'|} = \begin{cases} \dfrac{1}{r - r'}, & r > r' \\ \dfrac{1}{r' - r}, & r < r' \end{cases}$$

$r > r'$: $\quad \frac{1}{r - r'} = \frac{1}{r\left(1 - \frac{r'}{r}\right)} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \dots\right) =$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}} .$$

Zugleich: $\quad g_l(r, r') = \frac{(r')^l}{r^{l+1}} \quad (r > r') \quad (C_2 = 1).$

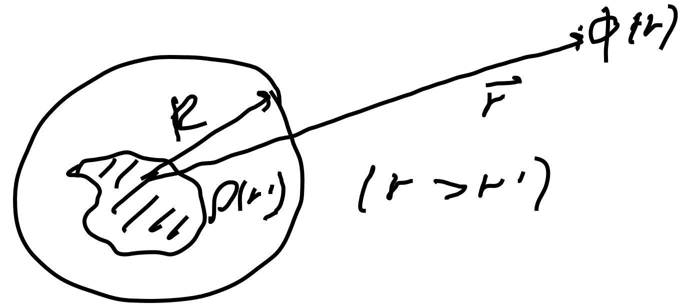
Analog: $r' > r$, $\quad g_l(r, r') = \frac{r^l}{(r')^{l+1}} \quad (r < r') \quad (C_1 = 1)$

$$G(\vec{r}, \vec{r}') = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{l} Y_{lm}^*(\theta', \varphi')\, Y_{lm}(\theta, \varphi) = \frac{1}{|\vec{r} - \vec{r}'|} .$$

$$r_< = \begin{cases} r, & r < r' \\ r', & r' < r \end{cases},$$

$$r_> = \begin{cases} r, & r > r' \\ r', & r' > r \end{cases} .$$

Multipolentwicklung



$(r > r')$

$$\nabla^2 \Phi(\bar{r}) = -\frac{1}{\varepsilon_0} \rho(\bar{r})$$

$$\rho(\bar{r}) = \int_{V'} \delta(\bar{r} - \bar{r}') \rho(\bar{r}') \, d^3\bar{r}'$$

$$\Phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} G(\bar{r}, \bar{r}') \rho(\bar{r}') \, d^3\bar{r}' =$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \underbrace{\int_{V'} \rho(r', \theta', \varphi') \cdot (r')^l \, Y^*_{lm}(\theta', \varphi') \, r'^2 dr' \sin\theta' d\theta' d\varphi'}_{M_{lm}}$$