$$\nabla^2 \varphi(r,\theta,\varphi) = 0 \qquad \varphi(r,\theta,\varphi) = V(r)\, P(\theta)\, Q(\varphi)$$

$$Q(\varphi) = e^{\pm i m \varphi}, \quad m \in \mathbb{Z}.$$

$m = 0$: $\quad P(\theta) \mapsto P_\ell(\cos\theta), \quad \ell = 0, 1, 2, \dots$

$(x = \cos\theta)$: $\quad P_\ell(x) = \dfrac{1}{2^\ell \ell!} \dfrac{d^\ell}{dx^\ell}(x^2-1)^\ell.$

$$\int_{-1}^{1} P_{\ell'}(x)\, P_\ell(x)\, dx = 0, \quad \ell \neq \ell'.$$

$\ell = \ell'$: $\quad \displaystyle\int_{-1}^{1} (P_\ell(x))^2\, dx = \frac{2}{2\ell+1}.$

Vollständigkeit: $\quad F(x) = \displaystyle\sum_{\ell=0}^{\infty} f_\ell\, P_\ell(x)$

$$f_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} F(x)\, P_\ell(x)\, dx.$$

Radialanteil:

$$\frac{1}{V}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \ell(\ell+1), \quad \ell = 0, 1, 2, \ldots$$

$$V(r) = A r^{\ell} + B r^{-\ell-1}$$

Winkelanteil: $m \neq 0$. $\frac{1}{\sin^2\theta} \frac{\partial^2 \Phi}{\partial \varphi^2}\Big|_{\Phi = e^{im\varphi}} = -\frac{m^2}{\sin^2\theta} e^{im\varphi}$

$x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2) \frac{dP}{dx} \right] + \left(l(l+1) - \frac{m^2}{1-x^2} \right) P = 0.$$

Lösung: die zugeordnete Legendre-Funktion;

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x).$$

mithilfe von Rodrigues-Formel:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$

Endliche Lösung nur für $m = -l, -l+1, \dots, l-1, l$.

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x).$$

Orthogonalität: $\int_{-1}^{1} P_\ell^m(x)\, P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell'\ell}$.

Vollständigkeit: $F(x) = \sum_\ell f_\ell\, P_\ell^m(x)$.

Winkelanteil: $Q_m(\varphi) \cdot P_\ell^m(\cos\theta)$, $\ell = 0, 1, \dots$ -

$m = -\ell, -\ell+1, \dots, \ell-1, \ell$.

$$\int_0^{2\pi} Q_m(\varphi)\, Q_{m'}^*(\varphi)\, d\varphi = \int_0^{2\pi} e^{i(m-m')\varphi}\, d\varphi = |m \neq m'| =$$

$$= \frac{e^{i(m-m')\varphi}}{i(m-m')}\Bigg|_0^{2\pi} = \frac{1-1}{i(m-m')} = 0 .$$

$$\int_0^{2\pi} Q_m(\varphi)\, Q_m^*(\varphi)\, d\varphi = 2\pi .$$

Kugelflächenfunktionen.

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\varphi}.$$

$$Y_{l,-m}(\theta,\varphi) = (-1)^m\, Y_{lm}^*(\theta,\varphi),$$

weil $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$.

Normierung und Orthogonalität

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\; Y_{l'm'}^*(\theta,\varphi)\, Y_{lm}(\theta,\varphi) = \delta_{ll'}\cdot\delta_{mm'}.$$

Darstellung der δ-Funktion!

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\cdot\delta(\cos\theta-\cos\theta') =$$

$$= \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\varphi-\varphi').$$

Lösung der Laplace-Gleichung

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$

$$\Phi(r,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Phi_{\ell m}(r,\theta,\varphi).$$

$$\Phi_{\ell m}(r,\theta,\varphi)=\left(A_{\ell m}r^{\ell}+B_{\ell m}r^{-\ell-1}\right)Y_{\ell m}(\theta,\varphi).$$

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

$$\ell=0,1,\ldots$$

$$m=-\ell,-\ell+1,\ldots,\ell-1,\ell.$$

Gleichung für die Greensfunktion.

$$\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,').$$

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2}\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta') =$$

$$= \frac{1}{r^2\sin\theta}\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\theta-\theta').$$

$$\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta';\varphi')\,Y_{\ell m}(\theta,\varphi)$$

$$\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} G(\vec{r},\vec{r}\,') =$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta';\varphi')\,Y_{\ell m}(\theta,\varphi)$$