Greensfunktionen.

Laplace-Gleichung in Kugelkoordinaten.

$$\Phi(r, \theta, \varphi) = V(r)\, P(\theta)\, Q(\varphi).$$

$$Q(\varphi) = e^{\pm i m \varphi}, \quad m \in \mathbb{Z}.$$

$$\boxed{\nabla^2 \Phi = f_1(r) + f_2(\theta) + f_3(\varphi) \quad \text{– Trennung der Variablen möglich.}}$$

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P}{\partial\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right) P = 0$$

Variablenwechsel: $x = \cos\theta \mapsto \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} = -\frac{\partial}{\partial x}$

$$\sin\theta \frac{\partial P}{\partial\theta} = -(1-x^2)\frac{\partial P}{\partial x}.$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\lambda - \frac{m^2}{1-x^2}\right] P = 0$$

$m = 0.$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \lambda P = 0.$$

Physikalische Lösung: $(x \Leftarrow\!\!= \cos\theta)$

$-1 \le x \le 1 \rightarrow$ Lösung eindeutig, endlich und stetig.

Suche
$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$$
$$P'(x) = \sum_{j=0}^{\infty} a_j (j+\alpha) x^{j+\alpha-1}$$
$$P''(x) = \sum_{j=0}^{\infty} a_j (j+\alpha)(j+\alpha-1) x^{j+\alpha-2}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] = (1-x^2)P'' - 2xP'$$

$$\sum_{j=0}^{\infty} a_j(j+\alpha)(j+\alpha-1)(1-x^2)x^{j+\alpha-2} - \sum_{j=0}^{\infty} 2a_j(j+\alpha)x^{j+\alpha} + \lambda\sum_{j=0}^{\infty} a_j x^{j+\alpha} = 0$$

$j-2 \longmapsto j$

$a_j \longmapsto a_{j+2}$

$j+\alpha \longmapsto j+2+\alpha$

$$\sum_{j=0}^{\infty}\left\{a_{j+2}(j+\alpha+2)(\alpha+j+1)\underbrace{-a_j(j+\alpha)(j+\alpha-1)-2a_j(j+\alpha)}_{-a(j+\alpha)(j+\alpha+1)}+\lambda a_j\right\}x^{j+\alpha}=0$$

$$j_{\text{alt}}=0 \longmapsto j=-2 \quad (j+2=0) \qquad a_0\,\alpha(\alpha-1)=0 \qquad (x^{\alpha-2})$$

$$j=-1: \quad (j+2=1) \qquad a_1\,\alpha(\alpha+1)=0 \qquad (x^{\alpha-1}).$$

$$a_{j+2}(j+\alpha+2)(j+\alpha+1)=\left[(j+\alpha)(j+\alpha+1)-\lambda\right]a_j$$

$$\alpha=0: \qquad \boxed{a_{j+2}=\frac{j(j+1)-\lambda}{(j+1)(j+2)}\,a_j}$$

a) $a_0\neq 0,\ a_1=0 \longmapsto$ gerade Potenzen von x

b) $a_0=0,\ a_1\neq 0 \longmapsto$ ungerade Potenzen von x

Endlichkeit: $j=\ell \Rightarrow \ell(\ell+1)-\lambda=0$

$$\lambda=\ell(\ell+1), \quad \ell=0,1,2\ldots$$

Legendre - Polinome $P_\ell(x)$

Normierung: $P_\ell(1) \overset{!}{=} 1$ → fixiert a_0 bzw. a_1.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

...

Rodrigues-Formel:

$$P_\ell(x) = \frac{1}{2^\ell \cdot \ell!} \frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell.$$

Wichtige Eigenschaften.

1) Orthogonalität:

$$\int_{-1}^{1} P_{l'}(x) \cdot P_l(x)\, dx = 0, \quad \text{für } l' \neq l.$$

$$l = l': \quad \int_{-1}^{1} \left(P_l(x)\right)^2 dx = \frac{2}{2l+1}.$$

2) Vollständigkeit: $\forall\, F(x), \ (-1 \leq x \leq 1)$

$$F(x) = \sum_{l=0}^{\infty} f_l\, P_l(x).$$

$$f_l = \frac{2l+1}{2} \int_{-1}^{1} F(x) \cdot P_l(x)\, dx.$$

Radial-Anteil der Laplace-Gleichung.

$$\lambda = \ell(\ell+1), \qquad \ell = 0, 1, \ldots$$

$V(r)$:
$$\frac{1}{V}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \ell(\ell+1)$$

$$2r\frac{dV}{dr} + r^2 \frac{d^2V}{dr^2} = \ell(\ell+1)V$$

$$V(r) = \frac{u(r)}{r}.$$

$$\frac{dV}{dr} = \frac{u'}{r} - \frac{u}{r^2};$$

$$\frac{d^2V}{dr^2} = \frac{u''}{r} - \frac{2u'}{r^2} + 2\frac{u}{r^3}$$

$$2r\left(\frac{u'}{r} - \frac{u}{r^2}\right) + r^2\left(\frac{u''}{r} - 2\frac{u'}{r^2} + 2\frac{u}{r^3}\right) = \ell(\ell+1)\frac{u}{r}$$

$$\underline{2u'} - 2\frac{u}{r} + ru'' - \underline{2u'} + 2\frac{u}{r} = \ell(\ell+1)\frac{u}{r}$$

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u = 0 \Rightarrow \underline{u = Ar^{\ell+1} + Br^{-\ell}}$$