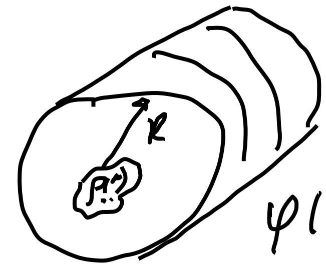
Multipolentwicklung und Green'sche Funktionen.

$$\nabla^2 \varphi(\vec r) = -\frac{1}{\varepsilon_0}\,\rho(\vec r).$$



$\varphi(R) = \text{const.}$

Hilfsproblem:

$$\nabla^2_{\vec r \vec r}\, G(\vec r, \vec r') = -4\pi\,\delta(\vec r - \vec r') \qquad \Big| + \text{Randbedingungen.}$$

$$G(r, r') \longmapsto \varphi(r)$$

$$\rho(\vec r) = \int_{V'} \delta(\vec r - \vec r')\,\rho(\vec r')\, d^3\vec r'$$

$$\int_{V'} \nabla^2_{\vec r \vec r}\, G(\vec r, \vec r')\,\rho(\vec r')\, d^3\vec r' = -4\pi \int_{V'} \delta(\vec r - \vec r')\,\rho(\vec r')\, d^3\vec r' = -4\pi\rho(\vec r)$$

$$\nabla^2_{\vec r \vec r} \int_{V'} G(\vec r, \vec r')\,\rho(\vec r')\, d^3\vec r' = -4\pi\rho(r)$$

$$\varphi(\vec r) = \frac{1}{4\pi\varepsilon_0} \int_{V} G(\vec r, \vec r')\,\rho(\vec r')\, d^3\vec r'.$$

Suche nach $G(\bar{r}, \bar{r}')$.

$$\nabla_{r}^{2}\, G(\bar{r}, \bar{r}') = -4\pi\, \delta(\bar{r} - \bar{r}').$$

Freie Randbedingungen

$\varphi(r \to \infty) = 0$

$$G(\bar{r}, \bar{r}') = \frac{1}{|\bar{r} - \bar{r}'|}.$$

$$\bar{r} \neq \bar{r}' : \quad \delta(\bar{r} - \bar{r}') = 0.$$

$$\nabla_{r}^{2}\, \varphi(\bar{r}) = 0 \quad - \text{ Laplace-Gleichung}$$

Kugelkoordinaten.

$$\nabla^2 \varphi = \left\{ \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} \varphi(r, \theta, \varphi) = 0$$

$$\varphi(r, \theta, \varphi) = V(r) \cdot P(\theta)\, Q(\varphi)$$

Trennung der Variablen

$$\Phi(r,\theta,\varphi) = V(r)\cdot P(\theta)\cdot Q(\varphi)$$

$$\nabla^2\Phi = P\cdot Q\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{V\cdot Q}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{VP}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$$\times\frac{r^2}{V\cdot P\cdot Q}\;\Bigg|\quad \frac{1}{V}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{Q\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$$F_1(r) + F_2(\theta,\varphi) = 0$$

$$F_1(r) = \lambda, \quad F_2(\theta,\varphi) = -\lambda.$$

$$\frac{1}{V}\partial_r\left(r^2\frac{\partial V}{\partial r}\right) = \lambda$$ – der Radialanteil

$$\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{Q\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = -\lambda$$ – der Winkelanteil.

Winkelanteil.

$$\times \sin^2\theta \Big| \qquad \underbrace{\frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\lambda\sin^2\theta}_{\theta\text{- Anteil}} = \underbrace{-\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2}}_{\varphi\text{ - Anteil}}.$$

φ - Anteil.

$$Q(\varphi+2\pi) = Q(\varphi).$$

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2} = -m^2 \quad \Rightarrow \quad Q(\varphi) = e^{\pm im\varphi}, \quad m \in \mathbb{Z}.$$

$$\frac{\partial Q}{\partial\varphi} = \pm im\, e^{\pm im\varphi}$$

$$\frac{\partial^2 Q}{\partial\varphi^2} = (\pm im)^2 e^{\pm im\varphi} = -m^2 e^{\pm im\varphi}$$

$$Q(\varphi) = C_1 e^{im\varphi} + C_2 e^{-im\varphi}$$

(C_1, C_2 - unbestimmt).

θ- Anteil

$$\frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\lambda\sin^2\theta=m^2$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)P=0$$

Variablentransformation $x=\cos\theta$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial(\cos\theta)}=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\qquad[d(\cos\theta)=-\sin\theta\, d\theta]$$

$$\sin\theta\frac{\partial P}{\partial\theta}=\sin\theta(-\sin\theta)\frac{\partial P}{\partial(\cos\theta)}=-\sin^2\theta\frac{\partial P}{\partial x}=$$

$$=-(1-\cos^2\theta)\frac{\partial P}{\partial x}=-(1-x^2)\frac{\partial P}{\partial x}.$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\lambda-\frac{m^2}{1-x^2}\right]P=0$$

zugeordnete Legendre'sche Diff.-Gleichung.