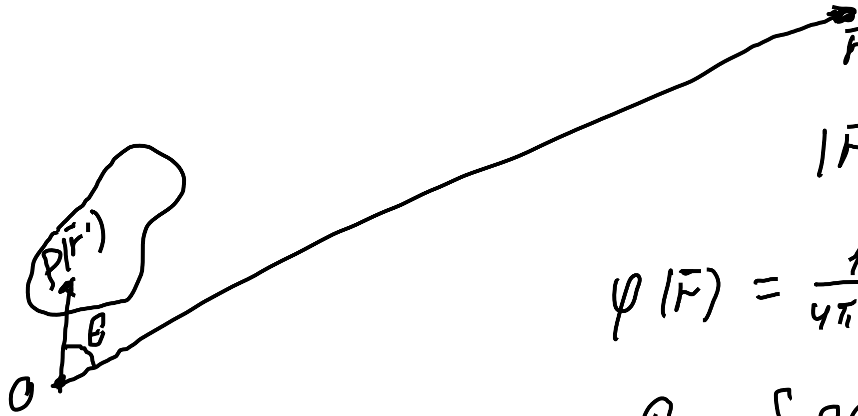
Multipolentwicklung.



$\vec{r}$ $\quad \varphi(\vec{r})$ - ?

$$|\vec{r}| >> |\vec{r}'|$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|} + \dots \text{ (Korrekturen)}$$

$$Q = \int_{V'} \rho(\vec{r}')\, d^3\vec{r}'$$

Exakte Lösung: $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}'$

$$\frac{1}{|\vec{r}-\vec{r}'|} \quad \left(|\vec{r}| >> |\vec{r}'| \Rightarrow \frac{|\vec{r}'|}{|\vec{r}|} << 1 \right).$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}')^{1/2}} = \frac{1}{r\left(1 + \frac{r'^2}{r^2} - 2\,\frac{r\cdot r'\cos\theta}{r^2}\right)^{1/2}} =$$

$$= \frac{1}{r}\cdot\frac{1}{\left[1 + \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta\right)\right]^{1/2}}.$$

Bezeichne $\varepsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta\right)$

$$\frac{1}{|\bar{r} - \bar{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \frac{\partial^n}{\partial \varepsilon^n}\left[\frac{1}{\sqrt{1+\varepsilon}}\right]\Bigg|_{\varepsilon=0}$$

$$\frac{\partial^n}{\partial \varepsilon^n}\left[\frac{1}{\sqrt{1+\varepsilon}}\right]\Bigg|_{\varepsilon=0} \equiv A_n$$

$$\frac{1}{|\bar{r} - \bar{r}'|} = \sum_{n=0}^{\infty} \frac{A_n}{n!} \frac{(r')^n}{r^{n+1}} \left(\frac{r'}{r} - 2\cos\theta\right)^n$$

Setze in $\varphi(\bar{r})$ ein:

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{A_n}{n!\, r^{n+1}} \int_{V'} \rho(\bar{r}')\,(r')^n \left(\frac{r'}{r} - 2\cos\theta\right)^n d^3\bar{r}' = \sum_{n=0}^{\infty} \varphi_n(\bar{r})$$

Multipolentwicklung

$$n=1: \quad A_1 = \left(-\frac{1}{2}\frac{1}{(1+\varepsilon)^{3/2}}\right)\Bigg|_{\varepsilon=0} = -\frac{1}{2}.$$

$$\varphi_1(\bar r) = \left(-\frac{1}{2}\right)\frac{1}{4\pi\varepsilon_0 r^2}\int_{V'} \rho(\bar r')\cdot r'\left(\frac{r'}{r} - 2\cos\theta\right) d^3\bar r' =$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\int_{V'}\rho(\bar r')\cdot r'\cos\theta\, d^3\bar r' - \frac{1}{8\pi\varepsilon_0}\int_{V'}\frac{\rho(\bar r')\, r'^2}{r^3}\, d^3\bar r'$$

$$\frac{1}{|\bar r - \bar r'|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\bar r'\cdot\nabla_r)^n\left(\frac{1}{|\bar r|}\right)$$ – unabhängig vom Koordinatensystem.

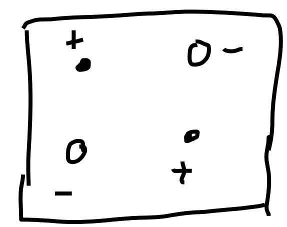
Kartesische Koordinaten:

$$\frac{1}{|\bar r - \bar r'|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\sum_{i=1}^{3} x_i'\,\partial_i\right)^n\left(\frac{1}{\sqrt{\sum_{i=1}^{3} x_i^2}}\right) =$$

$$= \frac{1}{r} + \frac{\sum_{i=1}^{3} x_i' x_i}{\left(\sum_{i=1}^{3} x_i^2\right)^{3/2}} + \dots = \frac{1}{r} + \frac{\bar r\cdot\bar r'}{r^3} + \dots$$

$$\varphi(\vec r) \approx \frac{1}{4\pi\varepsilon_0 r}\int_{V'}\rho(\vec r')\,d^3r' + \frac{1}{4\pi\varepsilon_0 r^3}\int_{V'}\rho(\vec r')(\vec r'\cdot\vec r)\,d^3r' =$$

$$= \frac{1}{4\pi\varepsilon_0 r}\underbrace{\int_{V'}\rho(\vec r')\,d^3r'}_{Q} + \frac{\vec r}{4\pi\varepsilon_0 r^3}\cdot\underbrace{\int_{V'}\rho(\vec r')\,\vec r'\,d^3r'}_{\vec p}$$

(+  + ...)