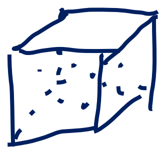
Ladungsverteilung.

δ - Funktion.

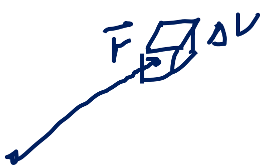
V



$\cdot \vec r'$

$$\sum_{\{i \in V\}} q_i = Q = \int_V \rho(\vec r)\, dV$$

$$V \longrightarrow \Delta V \qquad \int_V \rho(r)\, dV \longmapsto \rho(r) \cdot \Delta V$$



$$\rho(\vec r) = \lim_{\Delta V \to 0} \frac{Q(\vec r, \Delta V)}{\Delta V}$$

Ladungsdichte einer Punktladung.

$$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$Q = 1 \quad (\vec{r} = 0)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{r}}{|r|^3}.$$

$$\rho(\vec{r}) = \varepsilon_0 \cdot (\nabla \cdot \vec{E})$$

$$E_r = \frac{1}{4\pi\varepsilon} \frac{1}{r^2}, \quad E_\theta = 0, \quad E_\varphi = 0$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 E_r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{4\pi\varepsilon_0 r^2}\right) = \frac{1}{r^2} \frac{1}{4\pi\varepsilon_0} \frac{\partial}{\partial r}(1) = 0.$$

$$\oint_A \vec{E} \cdot \vec{n} \, dA = \frac{Q}{\varepsilon_0} \qquad (Q = 1)$$

A - Kugeloberfläche, Radius R.

$$\oint_A \vec{E} \cdot \vec{n} \, dA = \frac{1}{4\pi\varepsilon_0} \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \cdot R^2 \cdot \frac{1}{R^2} = \frac{1}{4\pi\varepsilon_0} \cdot 4\pi = \frac{1}{\varepsilon_0} \Rightarrow Q = 1.$$

Definiere: $\rho(\bar{r}) = \begin{cases} 0, & \text{für } \bar{r} \neq 0 \\ \text{singulär} & \text{für } \bar{r} = 0 \end{cases}$

$$\int_0^R \rho(\bar{r}) \cdot 4\pi r^2 \, dr = 1$$ – Gesamtladung $Q = 1$.

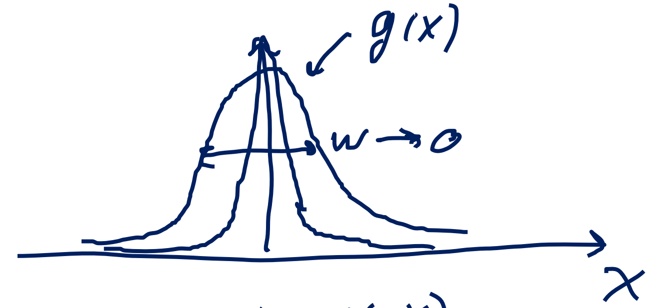
$$\rho(\bar{r}) \equiv \delta(\bar{r})$$

$$\int_0^R \delta(r) \cdot 4\pi r^2 \, dr = 1$$

$$\rho(\bar{r}) = \delta(\bar{r}) = \frac{1}{4\pi} \nabla \cdot \left(\frac{\bar{r}}{|r|^3} \right)$$

δ-Funktion in 1D.

$$\delta(x-x') = \begin{cases} 0, & x \neq x' \\ „\infty", & x = x' \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(x-x') f(x')\, dx' = f(x)$$

$f(x) = 1$: $\quad \displaystyle\int_{-\infty}^{\infty} \delta(x-x')\, dx' = 1.$

$$g(x) = g(-x)$$

$$\int_{-\infty}^{\infty} g(x)\, dx \stackrel{!}{=} 1.$$

Bsp: $\quad g_\alpha(x) = \dfrac{1}{\sqrt{2\pi}\,\alpha}\, e^{-\frac{x^2}{\alpha^2}}$

$$\delta(x) = \lim_{\alpha \to 0} g_\alpha(x).$$

$$\frac{d}{dx}\delta(x-x') = \ ?$$

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx'}\delta(x-x')\right) f(x')\, dx' = f(x')\,\delta(x-x')\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-x') \left(\frac{d}{dx'} f(x')\right) dx' =$$

$$= -f'(x)$$

$$\frac{d}{dx}\delta(x-x') = -\frac{d}{dx'}\delta(x-x')$$

$\delta(h(x))$:

Bsp. $x \to p$ - Impuls

$E_p = \frac{p^2}{2m}$

Energieerhaltung: $\vec{p} \to \vec{p}\,'$

$E_{p'} = E_p$: $\delta\left(\frac{p'^2}{2m} - \frac{p^2}{2m}\right) =$

$= \delta(h(p') - h(p))$

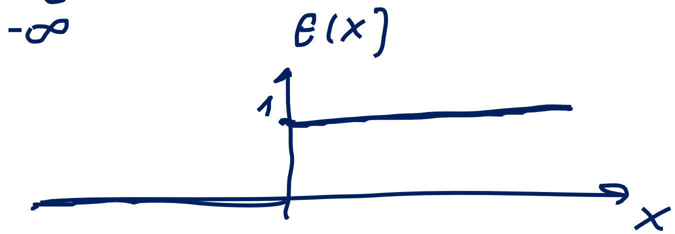
$$\int_{-\infty}^{\infty} \delta(h(x) - h(x'))\, f(x')\, dx' = \left| \begin{array}{l} x = x(h) \\ dx = \frac{dx}{dh} dh \end{array} \right| =$$

$$= \int_{-\infty}^{\infty} \delta(h - h')\, f(x'(h'))\, \frac{dx'}{dh'}\, dh' =$$

$$= \int_{-\infty}^{\infty} \frac{1}{(dh'/dx')}\, \delta(h - h')\, f(h')\, dh' = \frac{f(h)}{(dh/dx)},$$

$$\int_{-\infty}^{\infty} \delta\left(\frac{p^2}{2m} - \frac{p'^2}{2m}\right) f(p')\, dp' = f(p) \cdot \frac{1}{\underbrace{\frac{d}{dp}\left(\frac{p^2}{2m}\right)}_{= \frac{p}{m}}} = f(p) \cdot \frac{m}{p}.$$

$$\int_{-\infty}^{x} \delta(x')\,dx' = \theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ x = 0: & \text{zusätzliche Definition.} \end{cases}$$



Mehrdimensionale δ-Funktionen.

$D = 2$: $\quad \delta(\bar r - \bar r') = \delta(x - x')\,\delta(y - y')$

$D = 3$: $\quad \delta(\bar r - \bar r') = \delta(x - x')\,\delta(y - y')\,\delta(z - z')$.

$\cdots \qquad\qquad \cdots$

$D = 3$. Kugelkoordinaten r, θ, φ:

$$\delta(x - x')\,\delta(y - y')\,\delta(z - z')\,dx\,dy\,dz = \delta(\bar r - \bar r') \cdot r^2 \sin\theta\,d\theta\,d\varphi\,dr$$

$$\delta(\bar r - \bar r') = \frac{1}{r^2}\,\delta(r - r')\,\frac{1}{\sin\theta}\,\delta(\theta - \theta') \cdot \delta(\varphi - \varphi').$$

$$J = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| = \frac{1}{r^2 \sin\theta}.$$