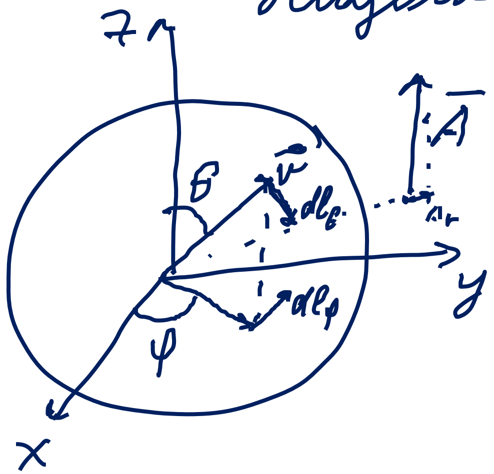
Kugelkoordinaten (r, θ, φ).



$$dl_r = dr$$

$$dl_\theta = r\, d\theta$$

$$dl_\varphi = r \sin\theta\, d\varphi$$

$$dV = r^2 \sin\theta\, d\theta\, d\varphi\, dr$$

$$dS_r = dl_\theta\, dl_\varphi = r^2 \sin\theta\, d\theta\, d\varphi$$

$$dS_\theta = dl_r\, dl_\varphi = r \sin\theta\, d\varphi\, dr$$

$$dS_\varphi = dl_r\, dl_\theta = r\, d\theta\, dr$$

$$\nabla = \begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\varphi \end{pmatrix}.$$

$$\vec{A} = (A_r, A_\theta, A_\varphi)$$

$$\nabla\cdot\vec{A} = \mathrm{div}\,\vec{A} = \frac{1}{r^2}\partial_r(r^2 A_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\theta) + \\ + \frac{1}{r\sin\theta}\partial_\varphi A_\varphi$$

$$(\vec{A}\cdot\vec{n})\,dS = (A_r\, dS_r)\big|_{r+dr} - (A_r\, dS_r)\big|_r + (A_\theta\, dS_\theta)\Big|_\theta^{\theta+d\theta} + \\ + (A_\varphi\, dS_\varphi)\Big|_\varphi^{\varphi+d\varphi} =$$

$$= \frac{\partial}{\partial r}\left(A_r\, r^2 \sin\theta\, d\theta\, d\varphi\right)\cdot dr + \partial_\theta\left(A_\theta \cdot r \sin\theta\, d\varphi\, dr\right) d\theta +$$

$$+ \partial_\varphi\left(A_\varphi\, r\, d\theta\, dr\right) d\varphi =$$

$$= \frac{\partial}{\partial r}\left(A_r r^2\right) \sin\theta\, d\theta\, d\varphi\, dr + \partial_\theta\left(\sin\theta\, A_\theta\right) r\, d\varphi\, dr\, d\theta +$$

$$+ \left(\partial_\varphi A_\varphi\right) r\, d\theta\, dr\, d\varphi =$$

$$= \underbrace{\left\{ \frac{1}{r^2}\partial_r\left(A_r r^2\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\partial_\varphi A_\varphi \right\}}_{\operatorname{div}\vec{A}} \underbrace{r^2 \sin\theta\, d\theta\, d\varphi\, dr}_{dV}$$

Laplace-Operator

$$\nabla^2 f(\vec{r}) = \text{div grad } f = \left\{ \frac{1}{r^2} \partial_r \left(r^2 \partial_r \right) + \right.$$

$$+ \frac{1}{r \sin\theta} \partial_\theta \left(\sin\theta \frac{1}{r} \partial_\theta \right) + \frac{1}{r \sin\theta} \partial_\varphi \left(\frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi} \right) \Big\} f(\vec{r}) =$$

$$= \left\{ \frac{1}{r^2} \partial_r \left(r^2 \partial_r \right) + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\} f(\vec{r}).$$

Rotation.

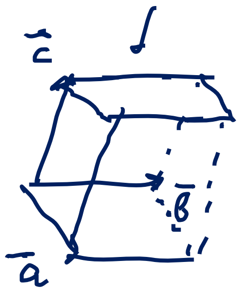
$$(\text{rot} A)_r = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \, A_\varphi \right) - \frac{\partial A_\theta}{\partial\varphi} \right]$$

$$(\text{rot} A)_\theta = \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial\varphi} - \frac{1}{r} \frac{\partial}{\partial r} \left(r A_\varphi \right)$$

$$(\text{rot} A)_\varphi = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial A_r}{\partial\theta} \right].$$

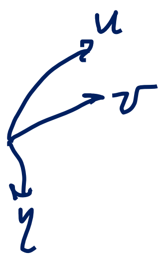
$$\int_V f(x,y,z)\,dx\,dy\,dz = \int_V f(r,\theta,\varphi)\, r^2 \sin\theta \, d\theta \, d\varphi \, dr = \int_V f(\vec{r})\, dV$$

$$V = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$



$$dV = d\vec{x}\cdot(d\vec{y}\times d\vec{z}) = \begin{vmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & dz \end{vmatrix} = dx\,dy\,dz$$

(u, v, η); $u = u(x,y,z)$, $v = v(x,y,z)$, $\eta = \eta(x,y,z)$



$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz$$

$$\begin{pmatrix} du \\ dv \\ d\eta \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x \eta & \partial_y \eta & \partial_z \eta \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

$$dV = du \cdot (dv \times d\eta) = \begin{vmatrix} \partial_x u \, dx & \partial_y u \, dy & \partial_z u \, dz \\ \partial_x v \, dx & \partial_y v \, dy & \partial_z v \, dz \\ \partial_x \eta \, dx & \partial_y \eta \, dy & \partial_z \eta \, dz \end{vmatrix} =$$

$$= \begin{vmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x \eta & \partial_y \eta & \partial_z \eta \end{vmatrix} dx\,dy\,dz = \left| \frac{\partial(u, v, \eta)}{\partial(x, y, z)} \right| dx\,dy\,dz$$

$\left| \frac{\partial(u,v,\eta)}{\partial(x,y,z)} \right|$ – Jacobi-Determinante

$$\left(\frac{\partial(u,v,\eta)}{\partial(x,y,z)} \right)^{-1} = \left(\frac{\partial(x,y,z)}{\partial(u,v,\eta)} \right)$$

$$\int_V f(\vec{r})\,dV = \int f(x,y,z)\,dx\,dy\,dz = \int f(u,v,\eta) \left| \frac{\partial(x,y,z)}{\partial(u,v,\eta)} \right| du\,dv\,d\eta$$

$x(u,v,\eta)$

$y(u,v,\eta)$

$z(u,v,\eta)$

Kartesische - Kugelkoordinaten

$$dV = dx\,dy\,dz = r^2 dr \sin\theta\, d\theta\, d\varphi = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| dr\, d\theta\, d\varphi$$

$$x(r,\theta,\varphi) = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

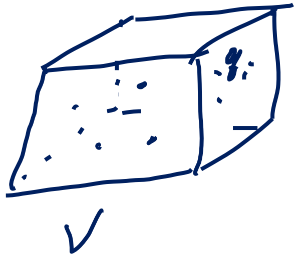
$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = \begin{vmatrix} \partial_r x & \partial_\theta x & \partial_\varphi x \\ \partial_r y & \partial_\theta y & \partial_\varphi y \\ \partial_r z & \partial_\theta z & \partial_\varphi z \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\varphi + r^2\cos\theta\sin\theta\sin^2\varphi\right] +$$

$$+ r\sin\theta\left[r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi\right] =$$

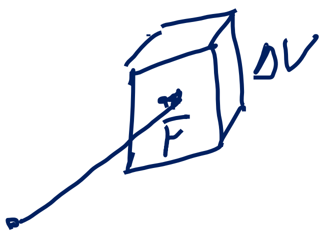
$$= r^2\cos^2\theta\sin\theta + r^2\sin^2\theta\sin\theta = r^2\sin\theta\,.$$

Ladungsdichte



Ladung im Volumen V: $Q = \sum_i q_i$

$$Q = \int_V \rho(\vec{r})\, dV = \sum_i q_i$$



$$\Delta Q = \rho(\vec{r}) \cdot \Delta V \quad \rightarrow \quad \rho(\vec{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q(\vec{r}, \Delta V)}{\Delta V}.$$