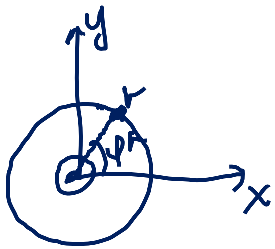
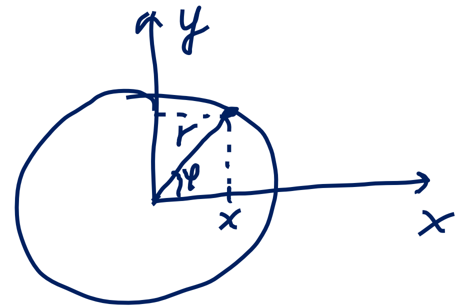
Krümmlinige Koordinaten

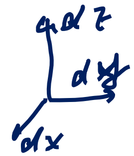
Zylindrisches Koordinatensystem



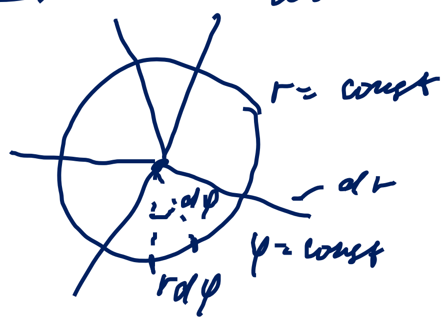
$$z \longrightarrow z$$
$$x \longrightarrow r\cos\varphi$$
$$y \longrightarrow r\sin\varphi$$



$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$



$$dz \longrightarrow dz$$



$$dl_r = dr$$
$$dl_\varphi = r\,d\varphi$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial l_r} \\ \frac{\partial}{\partial l_\varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \varphi} \end{pmatrix}$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

$$\vec{A} = (A_r, A_\varphi, A_z)$$

$$\operatorname{div} \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}.$$

$\operatorname{rot} \vec{A}$:

$$(\nabla \times \vec{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}$$

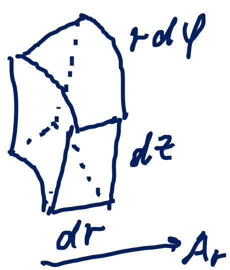
$$(\nabla \times \vec{A})_\varphi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \vec{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}$$

$\operatorname{div} \vec{A}$: Gauß'scher Satz.

$$d\Phi = d\left[\oint_{S=\partial V} (\vec{A}\cdot\vec{n})\cdot dS\right] = (\nabla\cdot\vec{A})\cdot dV$$

$dV =$



$$dS_r = r\,d\varphi\,dz$$

$$dS_\varphi = dr\,dz$$

$$dS_z = r\,dr\,d\varphi$$

$$dV = r\,dr\,d\varphi\,dz$$

$$d\Phi = \left(A_r\, r\, d\varphi\, dz\right)\Big|_{r+dr} - \left(A_r\, r\, d\varphi\, dz\right)\Big|_{r} +$$

$$+ \left(A_\varphi\, dr\, dz\right)\Big|_{\varphi+d\varphi} - \left(A_\varphi\, dr\, dz\right)\Big|_{\varphi} +$$

$$+ \left(A_z\, r\, dr\, d\varphi\right)\Big|_{z+dz} - \left(A_z\, r\, dr\, d\varphi\right)\Big|_{z} =$$

$$= \left(A_r(r+dr)(r+dr) - A_r(r)\cdot r\right) d\varphi\, dz +$$

$$+ \left(A_\varphi(\varphi+d\varphi) - A_\varphi(\varphi)\right) dr\, dz + \left(A_z(z+dz) - A_z(z)\right) r\, dr\, d\varphi =$$

$$= \left[\left(A_r(r) + \frac{\partial A_r}{\partial r} dr\right)(r+dr) - A_r(r)\cdot r\right] d\varphi\, dz +$$

$$+ \frac{\partial A_\varphi}{\partial \varphi} d\varphi\, dr\, dz + \frac{\partial A_z}{\partial z} r\, dr\, d\varphi\, dz =$$

$$= \left[\underline{A_r(r)\cdot r} + A_r(r)\, dr + \frac{\partial A_r}{\partial r} r\, dr + \frac{\partial A_r}{\partial r} dr^2 - \underline{A_r(r)\cdot r}\right] d\varphi\, dz +$$

$$+ \frac{\partial A_\varphi}{\partial \varphi} dr\, d\varphi\, dz + \frac{\partial A_z}{\partial z} r\, dr\, d\varphi\, dz =$$

$$= \frac{\partial}{\partial r}\left(r\cdot A_r\right) dr\, d\varphi\, dz + \frac{\partial A_\varphi}{\partial \varphi} dr\, d\varphi\, dz + \frac{\partial A_z}{\partial z} r\, dr\, d\varphi\, dz$$

$$d\varphi = \frac{\partial}{\partial r}(rA_r)\,dr\,d\varphi\,dz + \left(\frac{\partial}{\partial \varphi}A_\varphi\right)dr\,d\varphi\,dz + \left(\frac{\partial}{\partial z}A_z\right)r\,dr\,d\varphi\,dz =$$

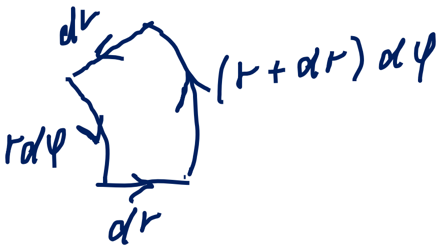
$$= (\nabla\cdot\bar{A})\cdot dV = (\nabla\cdot\bar{A})\cdot r\,dr\,d\varphi\,dz$$

$$(\nabla\cdot\bar{A}) = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}.$$

rot $\vec{A}$, aus dem Stokes'schen Satz.

$$d\left(\oint_{L=\partial S} \vec{A}\cdot d\vec{\ell}\right) = (\nabla\times\vec{A})\cdot\vec{n}\cdot dS \qquad S_z:\ \vec{n} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$dS = r\,dr\,d\varphi$$



$$d\oint \vec{A}\cdot d\vec{\ell} = \left[(rA_\varphi)\Big|_{r+dr} - (rA_\varphi)\Big|_{r}\right]d\varphi + \left[A_r\Big|_{\varphi} - A_r\Big|_{\varphi+d\varphi}\right]dr =$$

$$= \frac{\partial}{\partial r}(rA_\varphi)\,dr\,d\varphi - \frac{\partial A_r}{\partial \varphi}\,dr\,d\varphi =$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial\varphi}\right)\underbrace{r\,dr\,d\varphi}_{dS}$$

$$(\nabla\times\vec{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial\varphi}.$$

$$\Delta f = \nabla^2 f = \mathrm{div}(\mathrm{grad}\, f) = \left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right) + \frac{\partial^2}{\partial z^2}\right\}f =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$