Rotation (Wirbel) eines Feldes

$$\text{rot}\,\vec{E}(\vec{r}) = \nabla \times \vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = \hat{e}_x(\partial_y E_z - \partial_z E_y) +$$

$$+ \hat{e}_y(\partial_z E_x - \partial_x E_z) +$$

$$+ \hat{e}_z(\partial_x E_y - \partial_y E_x) =$$

$$= \varepsilon_{ijk}\,\hat{e}_i\,\partial_j E_k$$

$$\vec{E} = \begin{pmatrix} z \\ x \\ y \end{pmatrix} : \quad \nabla \times \vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} = \hat{e}_x \cdot 1 + \hat{e}_y \cdot 1 + \hat{e}_z \cdot 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\nabla \times \vec{E} \neq \underline{\vec{E} \times \nabla} - \text{Operator}$$

$$(\vec{E} \times \nabla)_i = \varepsilon_{ijk}\,E_j\,\partial_k$$

Partielle Ableitungen höherer Ordnung

$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y,z)}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}. \qquad \frac{\partial^2 f}{\partial y\,\partial x},\ \frac{\partial^2 f}{\partial z\,\partial x} \ldots$$

$$\frac{\partial^2 f}{\partial x_i\,\partial x_j};$$

$\frac{\partial^2 f}{\partial x\,\partial y}$ – gemischte Ableitung.

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x},$$ wenn kontinuierlich.

$$f(x,y,z) = xyz$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(x\cdot y\cdot z)\right) = \frac{\partial}{\partial x}(xz) = z$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(xyz)\right) = \frac{\partial}{\partial y}(yz) = z.$$

$$\nabla \times \nabla = \vec{0}.$$

$$\nabla \cdot \nabla = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \partial_x^2 + \partial_y^2 + \partial_z^2 =$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta -$$

– Laplace-Operator

$$\nabla \cdot \nabla f = \nabla \cdot (\nabla f) = \nabla \cdot (\operatorname{grad} f) = \operatorname{div} \operatorname{grad} f.$$

Taylor-Entwicklung Funktionen mehrerer Variablen

$f(x)$ (1. Variable)

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x-x_0) + \frac{1}{2}\frac{d^2 f(x_0)}{dx^2}(x-x_0)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad f^{(n)}(x_0) = \left.\frac{d^n f}{dx^n}\right|_{x=x_0}$$

$f(x,y,z)$: $\quad (x_0, y_0, z_0) \equiv \vec{r}_0$

$$f(x,y,z) = f(x_0,y_0,z_0) + \frac{\partial f(\vec{r}_0)}{\partial x}(x-x_0) + \frac{\partial f(\vec{r}_0)}{\partial y}(y-y_0) + \frac{\partial f(\vec{r}_0)}{\partial z}(z-z_0) +$$

$$+ \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + \frac{\partial^2 f}{\partial y^2}(y-y_0)^2 + \frac{\partial^2 f}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x-x_0)(y-y_0) +$$

$$+ \frac{\partial^2 f}{\partial y \partial x}(y-y_0)(x-x_0) + \frac{\partial^2 f}{\partial x \partial z} \dots \Big] + \dots$$

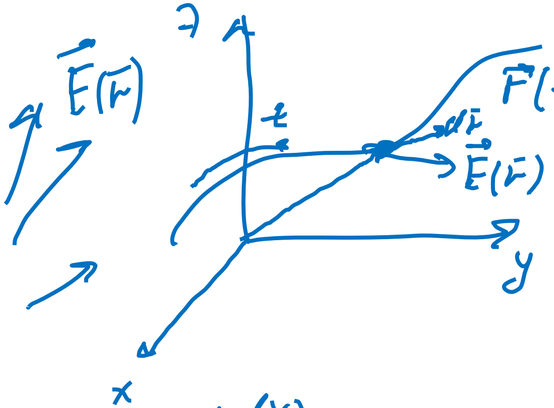
$$(x-x_0)\frac{\partial f}{\partial x} + (y-y_0)\frac{\partial f}{\partial y} + (z-z_0)\frac{\partial f}{\partial z} = \left[(\vec{r}-\vec{r}_0)\cdot\nabla\right] f$$

$$f(\vec{r}) = f(\vec{r}_0) + \left[(\vec{r}-\vec{r}_0)\cdot\nabla\right] f(\vec{r}_0) + \frac{1}{2}\left[(\vec{r}-\vec{r}_0)\cdot\nabla\right]^2 f(\vec{r}_0) + \ldots \text{ ש}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left[(\vec{r}-\vec{r}_0)\cdot\nabla\right]^n f(\vec{r}_0)$$

Integration der Felder

Linienintegral



$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

t – Parameter (lokale Koordinate)

$$d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$\int_C \vec{E}(\vec{r}) \cdot d\vec{r} = \int_C \left(E_x(\vec{r})\,dx + E_y(\vec{r})\,dy + E_z(\vec{r})\,dz \right) =$$

$$= \int_C \left(E_x(\vec{r}(t))\,dx + E_y(\vec{r}(t))\,dy + E_z(\vec{r}(t))\,dz \right) =$$

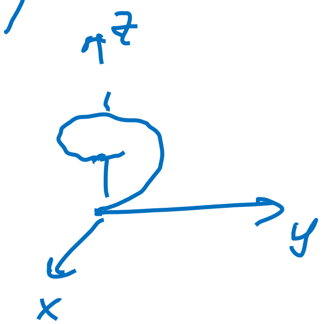
$$dx = \frac{dx}{dt}dt$$

$$dy = \frac{dy}{dt}dt$$

$$dz = \frac{dz}{dt}dt$$

$$= \int_{t_1}^{t_2} \left(E_x(\vec{r}(t))\frac{dx}{dt} + E_y(\vec{r}(t))\frac{dy}{dt} + E_z(\vec{r}(t))\frac{dz}{dt} \right) dt$$

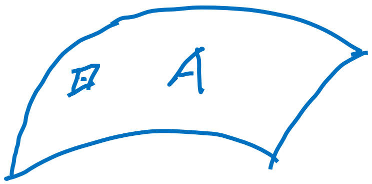
$$\vec{E}(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ e^{-z} \end{pmatrix}, \qquad C: \ \vec{r} = \begin{pmatrix} \cos t \\ \sin t \\ \frac{t}{2\pi} \end{pmatrix}, \quad 0 \le t \le 2\pi$$



$$\int_C \vec{E} \cdot d\vec{r} = \int_0^{2\pi} dt \left(1 \cdot (-\sin t) + e^{-\frac{t}{2\pi}} \cdot \frac{1}{2\pi} \right) =$$

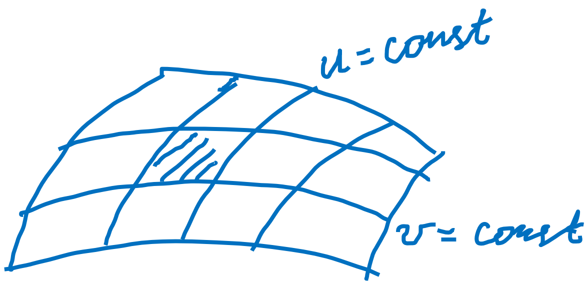
$$= -\int_0^{2\pi} \sin t \, dt + \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{t}{2\pi}} dt = 1 - \frac{1}{e}.$$

Oberflächenintegral



Skalare Funktion $f(\vec{F})$

$$\int_A f \cdot dA \,.$$

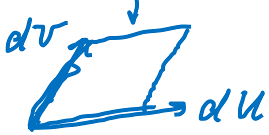


$$\vec{F} = \vec{F}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}.$$

$$f(\vec{F}) = f(\vec{F}(u,v)).$$

$$dA = |d\vec{u} \times d\vec{v}|$$

$$\int_A f \cdot dA = \int f(\vec{F}(u,v))\, |d\vec{u} \times d\vec{v}|$$



$$\begin{matrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{matrix} \longmapsto du,\; dv \,.$$