QM II Exam

22. 3. 2021

Suggestions for solutions of the problem

$\boxed{\#1}$ $\boxed{\text{Three-electron wave function}}$

Three different states, $|\psi_\alpha\rangle \equiv |\alpha\rangle$, $|\psi_\beta\rangle \equiv |\beta\rangle$, $|\psi_\gamma\rangle \equiv |\gamma\rangle$
$\Rightarrow$ Totally antisymmetric properly normalized state for the 3-electron configuration is:

$$|\psi_{\alpha\beta\gamma}\rangle = \frac{1}{\sqrt{3!}}\Big[|\alpha\rangle^{(1)}|\beta\rangle^{(2)}|\gamma\rangle^{(3)} + |\beta\rangle^{(1)}|\gamma\rangle^{(2)}|\alpha\rangle^{(3)} + $$
$$+ |\gamma\rangle^{(1)}|\alpha\rangle^{(2)}|\beta\rangle^{(3)} - |\gamma\rangle^{(1)}|\beta\rangle^{(2)}|\alpha\rangle^{(3)} - $$
$$- |\beta\rangle^{(1)}|\alpha\rangle^{(2)}|\gamma\rangle^{(3)} - |\alpha\rangle^{(1)}|\gamma\rangle^{(2)}|\beta\rangle^{(3)} \Big]$$

$$= \frac{1}{\sqrt{3!}} \begin{vmatrix} |\alpha\rangle^{(1)} & |\alpha\rangle^{(2)} & |\alpha\rangle^{(3)} \\ |\beta\rangle^{(1)} & |\beta\rangle^{(2)} & |\beta\rangle^{(3)} \\ |\gamma\rangle^{(1)} & |\gamma\rangle^{(2)} & |\gamma\rangle^{(3)} \end{vmatrix}$$

Slater determinant.

#2 Simultaneous diagonalizability of many-body (1-particle) observables

(a) $0 \overset{!}{=} [\hat{O}_1, \hat{O}_1'] = \sum_{i,j,k,l} O^{(1)}_{ij} O^{(1)\prime}_{kl} \underbrace{[\hat{c}_i^\dagger \hat{c}_j, \hat{c}_k^\dagger \hat{c}_l]}$

$$\hat{c}_i^\dagger [\hat{c}_j, \hat{c}_k^\dagger \hat{c}_l] + [\hat{c}_i^\dagger, \hat{c}_k^\dagger \hat{c}_l] \hat{c}_j$$

$$= \hat{c}_i^\dagger \{\hat{c}_j, \hat{c}_k^\dagger\} \hat{c}_l - \hat{c}_k^\dagger \{\hat{c}_i^\dagger, \hat{c}_l\} \hat{c}_j$$

$$= \hat{c}_i^\dagger \hat{c}_l \delta_{jk} - \hat{c}_k^\dagger \hat{c}_j \delta_{il}$$

with $[\hat{a}, \hat{b}\hat{c}] = \{\hat{a}, \hat{b}\}\hat{c} - \hat{b}\{\hat{a}, \hat{c}\}$

$$= \sum_{i,j,k} O^{(1)}_{ik} O^{(1)\prime}_{kl} \hat{c}_i^\dagger \hat{c}_l$$

$$- \sum_{i,j,k} O^{(1)}_{ij} O^{(1)\prime}_{ki} \hat{c}_k^\dagger \hat{c}_j$$

rename:
$j \to l$
$k \to i$
$i \to j$

$$= \sum_{i,j,k} [O^{(1)}, O^{(1)\prime}]_{il} \, \hat{c}_i^\dagger \hat{c}_l \overset{!}{=} 0$$

$\Rightarrow [O^{(1)}, O^{(1)\prime}]_{il} = 0$ The commutator of 1-particle matrix elements must vanish.

(b) This is the same for bosons, using

$$[\hat{a}, \hat{b}\hat{c}] = [\hat{a}, \hat{b}]\hat{c} + \hat{b}[\hat{a}, \hat{c}]$$

The relative sign comes from $[\hat{b}_i^\dagger, \hat{b}_l] = -1$.

$\boxed{\#3}$ $\boxed{\text{Quantitation of linear (phonon) chain}}$

(a) Orthonormality conditions for $k \neq k'$

$$\sum_{n=1}^{N} z_n^{k'*} z_n^{k} = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i (k-k')\frac{n}{N}}$$

$$= \frac{1-e^{2\pi i (k-k')(1+1/N)}}{1-e^{2\pi i (k-k')/N}}$$

geometric series, $\sum_{n=1}^{N} a^n = \frac{1-a^{N+1}}{1-a}$

$$= \frac{e^{2\pi i (k-k')}-1}{e^{2\pi i (k-k')/N}-1} = 0,$$

denominator is non-zero.

as $e^{2\pi i (k-k')} = 1$, as $k-k'$ integer.

for $k=k'$ $\frac{1}{N}\sum_{n=1}^{N} 1 = 1.$ ✓

Completeness relation:

$$\sum_{k=1}^{N} z_{n'}^{k*} z_n^{k} = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(\ell-\ell')} = \delta_{nn'}$$

(same calculation as above).

$$\hat{q}_n = \sum_{k=1}^{N} \sqrt{\frac{\hbar}{2m\omega_k}} \left(z_n^k \hat{b}_k + z_n^{k*} \hat{b}_k^\dagger \right)$$

$\cdot \frac{i}{m\omega_k}$ /

$$\hat{p}_n = -i\sum_{k=1}^{N} \sqrt{\frac{\hbar m \omega_k}{2}} \left(z_n^k \hat{b}_k - z_n^{k*} \hat{b}_k^\dagger \right)$$

$$\Rightarrow \quad \hat{q}_n + \frac{i}{m\omega_k}\hat{p}_n = \sum_{k=1}^{N} 2\sqrt{\frac{\hbar}{2m\omega_k}}\, z_n^k \hat{b}_k \quad \Big/ \sum_n z_n^{k'*}$$

$$\Rightarrow \quad \hat{b}_k = \sqrt{\frac{m\omega_k}{2\hbar}} \sum_n z_n^{k*} \left(\hat{q}_n + \frac{i}{m\omega_k}\hat{p}_n \right)$$

$$= \sum_n z_n^{k*} \left(\sqrt{\frac{m\omega_k}{2\hbar}}\, \hat{q}_n + i\sqrt{\frac{1}{2\hbar m\omega_k}}\, \hat{p}_n \right)$$

$$\hat{b}_k^\dagger = \sum_n z_n^{k} \left(\sqrt{\frac{m\omega_k}{2\hbar}}\, \hat{q}_n - i\sqrt{\frac{1}{2\hbar m\omega_k}}\, \hat{p}_n \right)$$

$$\Rightarrow [\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^+, \hat{b}_{k'}^+] = 0$$

$$[\hat{b}_k, \hat{b}_{k'}^+] = \sum_{n,n'} z_n^{k*} z_{n'}^{k'} \left\{ \left(\frac{-i}{2\hbar}\right) \sqrt{\frac{\omega_k}{\omega_{k'}}} \underbrace{[\hat{q}_n, \hat{p}_{n'}]}_{i\hbar\delta_{nn'}} + \left(\frac{i}{2\hbar}\right) \sqrt{\frac{\omega_{k'}}{\omega_k}} \underbrace{[\hat{p}_n, \hat{q}_{n'}]}_{-i\hbar\delta_{nn'}} \right.$$

$$= \frac{1}{2} \underbrace{\sum_n z_n^{k*} z_n^{k'}}_{\delta_{kk'}} \left(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}} \right) = \delta_{kk'} \quad \checkmark$$

(b) $$\sum_{n=1}^{N} \frac{\hat{p}_n^2}{2m} = \sum_{n,k,k'} \left(\frac{-1}{2\cancel{m}}\right) \cdot \frac{\hbar \cancel{m} \sqrt{\omega_k \omega_{k'}}}{2} \left(z_n^k \hat{b}_k - z_n^{k*} \hat{b}_k^+ \right) \left(z_n^{k'} \hat{b}_{k'} - z_n^{k'*} \hat{b}_{k'}^+ \right)$$

$$= \sum_{k=1}^{N} \frac{\hbar \omega_k}{4} \left(\hat{b}_k \hat{b}_k^+ + \hat{b}_k^+ \hat{b}_k \right) - \sum_{k=1}^{N} \frac{\hbar \sqrt{\omega_k \omega_{k'}}}{4} \left(\hat{b}_k \hat{b}_{-k} + \hat{b}_k^+ \hat{b}_{-k}^+ \right)$$

To proof: $\sum_{n=1}^{N} z_n^{k'} z_n^{k} = \sum_{n=1}^{\infty} z_n^{k'} z_n^{-k*} = \delta_{k',-k}$

(the other formula by complex conjugation)

$$z_{n\pm1}^{k} = \frac{1}{\sqrt{N}} e^{2\pi i \frac{k}{N}(n\pm1)} = e^{\pm 2\pi i \frac{k}{N}} z_n^*$$

$$\sum_{n=1}^{N} \frac{K}{2} (\hat{q}_{n+1} - \hat{q}_n)^2 = \frac{\hbar K}{4m} \sum_{n=1}^{N} \sum_{k,k'} \Big\{ \hat{b}_{k'} \hat{b}_k \left(e^{2\pi i \frac{k'}{N}} - 1\right) \left(e^{2\pi i k/N} - 1\right) z_n^{k'} z_n^{k} + \hat{b}_{k'} \hat{b}_k^+ \left(e^{2\pi i k'/N} - 1\right)\left(e^{-2\pi i k/N} - 1\right) z_n^{k'} z_n^{k*} + \hat{b}_{k'}^+ \hat{b}_k \left(e^{-2\pi i k'/N} - 1\right)\left(e^{+2\pi i k/N} - 1\right) z_n^{k'*} z_n^{k}$$

$$+ \hat b^{\dagger}_{k'} \hat b^{\dagger}_{k} \left(e^{-2\pi i k'/N} - 1\right)\left(e^{-2\pi i k/N} - 1\right) z_n^{k'*} z_n^{k*} \Big\} \frac{1}{\sqrt{\omega_k \omega_{k'}}}$$

$$= \sum_k \frac{\kappa \hbar}{4m} \underbrace{\underbrace{\left(e^{-2\pi i k/N} - 1\right)\left(e^{+2\pi i k/N} - 1\right)}_{2 - 2\cos\left(\frac{2\pi k}{N}\right)}}_{\equiv \frac{\hbar \omega_k^2}{4m}} \cdot \Big\{ \frac{1}{\omega_k}\left(\hat b_k \hat b_k^{\dagger} + \hat b_k^{\dagger} \hat b_k\right) + \frac{1}{\sqrt{\omega_k \omega_k}}\left(\hat b_k \hat b_{-k} + \hat b_k^{\dagger} \hat b_{-k}^{\dagger}\right) \Big\}$$

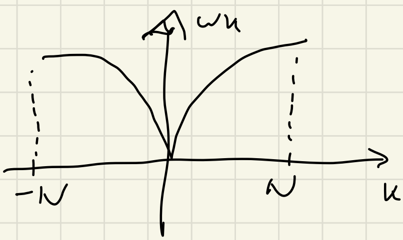
In order for the mixed terms $(\hat b_k \hat b_{-k}, \hat b_k^{\dagger} \hat b_{-k}^{\dagger})$ to cancel, we need:

$$\boxed{\omega_k^2 = \omega_{-k}^2 = \frac{2\kappa}{m}\left(1 - \cos\left(\frac{2\pi k}{N}\right)\right)}$$

$$\Rightarrow \boxed{\hat H} = \sum_{n=1}^{N} \frac{\hat p_n^2}{2m} + \frac{\kappa}{2} \sum_{n=1}^{N} (\hat q_{n+1} - \hat q_n)^2$$

$$= \sum_k \frac{\hbar \omega_k}{2} \left(\hat b_k \hat b_k^{\dagger} + \hat b_k^{\dagger} \hat b_k\right)$$

$$= \boxed{\sum_k \hbar \omega_k \left(\hat b_k^{\dagger} \hat b_k + \frac{1}{2}\right)} \quad \boxed{\omega_k = \sqrt{\frac{2\kappa}{m}\left(1 - \cos\left(\frac{2\pi k}{N}\right)\right)}}$$



(c) $\hat H = \hat H_0 + \hat V$, $\hat V = -\sum_n K_n \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left(\hat b_k z_n^k + \hat b_k^{\dagger} z_n^{k*}\right) = -\sum_k \left(F_k \hat b_k + F_k^* \hat b_k^{\dagger}\right)$

with $F_k = \sum_n K_n \sqrt{\frac{\hbar}{2m\omega_k}} z_n^k = \frac{1}{\sqrt{N}} \sqrt{\frac{\hbar}{2m\omega_k}} \sum_n K_n e^{2\pi i \frac{kn}{N}}$

$$\Rightarrow \hat H = \sum_k \left\{ \hbar \omega_k \left(\hat b_k^{\dagger} \hat b_k + \frac{1}{2}\right) - F_k \hat b_k - F_k^* \hat b_k^{\dagger} \right\}$$

Diagonalization by linear shift ("completion of square")

$$\hat{c}_k = \hat{b}_k - \alpha_k \,, \quad \hat{c}_k^+ = \hat{b}_k^+ - \alpha_k^* \implies [\hat{c}_k, \hat{c}_{k'}] = 0 \,,$$

$$[\hat{c}_k^+, \hat{c}_{k'}^+] = 0 \,, \quad [\hat{c}_k, \hat{c}_{k'}^+] = \delta_{kk'} \implies$$

$$\hat{H} = \sum_k{}' \Big[\hbar\omega_k \left(\hat{c}_k^+ \hat{c}_k + \tfrac{1}{2}\right) - \hat{c}_k \left(F_k \cancel{- \alpha_k^* \hbar\omega_k}\right)$$
$$- \hat{c}_k^+ \left(F_k^* \cancel{- \alpha_k \hbar\omega_k}\right) + \hbar\omega_k |\alpha_k|^2$$
$$- \alpha_k F_k - \alpha_k^* F_k^* \Big]$$

To remove linear terms choose: $\alpha_k = \frac{1}{\hbar\omega_k} F_k^*$

constant terms: $\hbar\omega_k |\alpha_k|^2 - \alpha_k F_k - \alpha_k^* F_k^*$

$$= - \hbar\omega_k |\alpha_k|^2 = - \frac{1}{\hbar\omega_k} |F_k|^2 .$$

We set: $$\Delta E = - \sum_k{}' \frac{1}{\hbar\omega_k} |F_k|^2$$

$$= - \sum_k \frac{1}{\cancel{\hbar}\omega_k} \frac{\cancel{\hbar}}{2m\omega_k} \sum_{n,n'} z_n^k z_{n'}^{k*} K_n K_{n'}$$

$$= - \sum_k \sum_{n,n'} \frac{K_n K_{n'}}{2m\omega_k^2} z_n^k z_{n'}^k$$

$$\implies \boxed{\hat{H} = \sum_k \hbar\omega_k \left(\hat{c}_k^+ \hat{c}_k + \tfrac{1}{2}\right) + \Delta E}$$

global lowering of all energy levels, but shape of spectrum is unchanged.

#4 $\boxed{\text{Canonical ensemble}}$

Probability distribution: $W_n = \frac{1}{Z} \exp[-\beta E_n]$

$$1 \overset{!}{=} \sum_n W_n = \frac{1}{Z} \sum_n \exp[-\beta E_n] \Rightarrow$$

$$\boxed{Z = \sum_n \exp[-\beta E_n]} \quad \Longrightarrow$$

$$-\frac{d}{d\beta} \ln Z = -\frac{1}{Z} \frac{d}{d\beta} Z = \frac{1}{Z} \left(-\frac{d}{d\beta}\right) \sum_n \exp[-\beta E_n]$$

$$= \frac{1}{Z} \sum_n E_n \exp[-\beta E_n] = \sum_n E_n W_n = \langle E \rangle \quad \boxed{1}$$

$$\frac{d^2}{d\beta^2} \ln Z = \frac{d}{d\beta}\left(\frac{1}{Z} \frac{d}{d\beta} Z\right) = -\frac{1}{Z^2}\left(\frac{d}{d\beta} Z\right)\left(\frac{d}{d\beta} Z\right)$$

$$+ \frac{1}{Z} \frac{d^2}{d\beta^2} Z = \frac{1}{Z} \sum_n E_n^2 \exp[-\beta E_n] -$$

$$\left(-\frac{1}{Z} \frac{d}{d\beta} Z\right)^2 = \langle E^2 \rangle - \langle E \rangle^2 \quad \boxed{2}$$

#5 $\boxed{\text{Canonical ensemble of harmonic oscillators}}$

Partition function: $Z = \sum_{n=0}^{\infty} \exp\left[-\beta \hbar\omega \left(n + \frac{1}{2}\right)\right]$

$$= e^{-\hbar\omega\beta/2} \sum_n \left(e^{-\hbar\omega\beta}\right)^n = \frac{e^{-\hbar\omega\beta/2}}{1 - e^{-\hbar\omega\beta}}$$

(geometric series). The zero point energy drops from the probability distribution:

$$W_n = \frac{1}{Z} e^{-\beta E_n} = \left(1 - e^{-\hbar\omega\beta}\right) \cancel{e^{\hbar\omega\beta/2}} \cdot$$

$$\exp[-\beta\hbar\omega n] \cancel{e^{-\hbar\omega\beta/2}} = e^{-n\hbar\beta\omega}\left(1 - e^{-\hbar\beta\omega}\right)$$

Define a modified partition function without the ground state energy, $Z \longmapsto Z = \dfrac{1}{1-\exp[-\beta\hbar\omega]}$

The physics behind is that all energies will be measured with respect to the new ground state energy $E_0 = \frac{1}{2}\hbar\omega$.

$$\Rightarrow \langle E \rangle = -\frac{d}{d\beta}\ln Z = \frac{d}{d\beta}\ln\left(1-e^{-\hbar\omega\beta}\right) = \frac{\hbar\omega}{e^{\hbar\omega\beta}-1}$$

This is the Bose distribution.

$$\langle E^2 \rangle = \frac{d^2}{d\beta^2}\ln Z + \langle E \rangle^2 = -\frac{d}{d\beta}\langle E \rangle + \langle E \rangle^2$$

$$= \frac{(-\hbar\omega)(-\hbar\omega)e^{\hbar\omega\beta}}{(e^{\hbar\omega\beta}-1)^2} + \frac{(\hbar\omega)^2}{(e^{\hbar\omega\beta}-1)^2} = (\hbar\omega)^2\frac{e^{\hbar\omega\beta}+1}{(e^{\hbar\omega\beta}-1)^2}$$

<u>Limits:</u> high temperature (classical) limit

$$\lim_{\omega\beta\to 0}\langle E \rangle = \frac{\hbar\omega}{\cancel{1}+\frac{\hbar\omega}{k_B T}+O((\omega\beta)^2)-\cancel{1}} = \frac{1}{\beta} = k_B T \quad \text{ideal gas.}$$

$$\lim_{\omega\beta\to\infty}\langle E \rangle = \hbar\omega\, e^{-\hbar\omega/k_B T} \quad \text{Boltzmann distribution}$$

The occupation probability is for $\omega\beta \to 0$ a flat distribution (all states have identical probabilities), for $\omega\beta\to\infty$ the occupation probability goes to zero for all states except for the ground state.

$\boxed{\#6}$ Time evolution of Klein-Gordon operator

LHS of the Schrödinger equation:

$$i \frac{\partial}{\partial t} \hat{\phi}(x) = \int \widetilde{dp} \left(\hat{a}_p e^{-ipx} - \hat{a}_p^\dagger e^{+ipx} \right) E_p$$

RHS of the Schrödinger equation:

$$[\hat{\phi}(x), \hat{H}] = \int \widetilde{dp} \int \widetilde{dq}\, E_q \Big(\overbrace{[\hat{a}_p, \hat{a}_q^\dagger \hat{a}_q]}^{(2\pi)^3 (2E)_q \delta^3(p-q)\, \hat{a}_q} e^{-ipx}$$

$$+ \underbrace{[\hat{a}_p^\dagger, \hat{a}_q^\dagger \hat{a}_q]}_{\hat{a}_q^\dagger (-)\cdot(2E_q)(2\pi)^3 \delta^3(p-q)} e^{+ipx} \Big)$$

$$= \int \widetilde{dp} \int \frac{d^3 q}{\cancel{(2\pi)^3 2E_q}} \cancel{(2E_q)} \cdot \cancel{(2\pi)^3} \delta^3(p-q) \Big(\hat{a}_p e^{-iqx}$$

$$- \hat{a}_p^\dagger e^{+iqx} \Big) E_p$$

$p = q \Rightarrow p^0 = q^0$ for momenta on the mass-shell

$$= \int \widetilde{dp}\, E_p \left(\hat{a}_p e^{-ipx} - \hat{a}_p^\dagger e^{+ipx} \right). \quad \checkmark$$

$\boxed{\#7}$ $\boxed{\text{System of two complex scalar fields}}$

(a) Equation of motion for ϕ from ϕ^*:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)} = \frac{\partial \mathcal{L}}{\partial \phi_i^*} \Rightarrow \partial_\mu \partial^\mu \phi_i = -m^2 \phi_i - \lambda^2 (\phi_k^* \phi_k)\phi_i$$

$$\Rightarrow \boxed{(\Box + m^2)\phi_i(x) = -\lambda^2 (\phi_k^* \phi_k)\phi_i}$$

The other e.o.m. is the Hermitian adjoint.

(b) $\phi_i \longmapsto \phi_i + i\varepsilon_a \frac{\sigma^a_{ij}}{2} \phi_j \Rightarrow$

$$\phi_i^* \longmapsto \phi_i^* + \phi_j^* \frac{(\sigma^a_{ij})^\dagger}{2}(-i)\varepsilon_a$$

$$= \phi_i^\dagger + \phi_j^\dagger \frac{(\sigma^a)_{ji}}{2}(-i)\varepsilon^a,$$ as the Pauli matrices are Hermitian.

$$\Rightarrow \phi_i^* \phi_i \longmapsto \phi_i^* \phi_i + i\varepsilon_a \cancel{\phi_i^* \frac{\sigma^a_{ij}}{2} \phi_j} - i\varepsilon_a \cancel{\phi_j^* \frac{\sigma^a_{ji}}{2}} \phi_i + \mathcal{O}(\varepsilon^2)$$

$$= \phi_i^* \phi_i + \mathcal{O}(\varepsilon^2)$$ invariant.

(Also applies for the finite [exponentiated] transformation,

$$\phi_i \longmapsto \exp\left[i\varepsilon_a \frac{\sigma^a}{2}\right]_{ij} \phi_i,$$

$$\phi_i^* \longmapsto \phi_j^* \exp\left[-i\varepsilon_a \frac{\sigma^a}{2}\right]_{ji}$$

Because the ε^a are constant, also $(\partial_\mu \phi_i^*)(\partial^\mu \phi_i)$ is invariant.

Noether currenti

$$\boxed{j^\mu(x)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta_\varepsilon \phi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)} \delta_\varepsilon \phi_i^*(x)$$

$$= \partial^\mu \phi_i^* \left(+i \frac{\sigma^a_{ij}}{2} \phi_j\right) + (\partial^\mu \phi_i)\left(-i \frac{\sigma^a_{ji}}{2} \phi_j^*\right)$$

$$\boxed{= -i\, \phi_i^* \left(\frac{\sigma^a}{2}\right)_{ij} \overleftrightarrow{\partial}^\mu \phi_j}$$

(c) $$\partial_\mu j^\mu = -i\, \partial_\mu \left[\phi^* \frac{\sigma^a}{2} \overleftrightarrow{\partial}^\mu \phi\right]$$

$$= -i\, \partial_\mu \left(-\partial^\mu \phi^* \frac{\sigma^a}{2} \phi + \phi^* \frac{\sigma^a}{2} \partial^\mu \phi\right)$$

$$= i\left(\cancel{\partial^\mu \phi^* \frac{\sigma^a}{2} \partial_\mu \phi} + \Box \phi^* \frac{\sigma^a}{2} \phi - \cancel{\partial^\mu \phi^* \frac{\sigma^a}{2} \partial_\mu \phi} - \phi^* \frac{\sigma^a}{2} \Box \phi\right)$$

$$\overset{\text{e.o.m.}}{=} i\left(\cancel{-m^2 \phi^* \frac{\sigma^a}{2} \phi} - \lambda^2 \cancel{\left(\phi^* \frac{\sigma^a}{2} \phi\right)}(\phi^* \phi) + \cancel{m^2 \phi^* \frac{\sigma^a}{2} \phi} + \lambda^2 \cancel{\left(\phi^* \frac{\sigma^a}{2} \phi\right)}(\phi^* \phi)\right)$$

$$= 0$$

$\square$

(d) $$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \partial^\nu \phi^* - g^{\mu\nu} \mathcal{L}$$

$$= (\partial^\mu \phi^*)(\partial^\nu \phi) + (\partial^\mu \phi)(\partial^\nu \phi)^* - g^{\mu\nu} (\partial_\sigma \phi^*)(\partial^\sigma \phi) + g^{\mu\nu} m^2 (\phi^* \phi) + \lambda^2 g^{\mu\nu} (\phi^* \phi)^2$$

$\boxed{\#8}$ $\boxed{\text{Properties of gamma matrices}}$

(a) Transposition of γ matrices:

$$\gamma^{0T} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}^T = \gamma^0 \,, \quad \gamma^{iT} = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & -\sigma^{iT} \\ +\sigma^{iT} & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & -\sigma^i \\ +\sigma^i & 0 \end{pmatrix} = -\gamma^i & \text{for } i = 1,3 \\ \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma^i & \text{for } i = 2 \end{cases}$$

$\Rightarrow \gamma^0, \gamma^2$ symm., γ^1, γ^3 antisymmetric.

(b) $\mathcal{C} = i\gamma^2\gamma^0$. Proof: $\mathcal{C}^{-1} = -\mathcal{C} \rightarrow$

$$-\mathcal{C}^2 = -\mathcal{C}\cdot\mathcal{C} = i\gamma^2\gamma^0(-i)\gamma^2\gamma^0 = \gamma^2\gamma^0\gamma^2\gamma^0$$

$$= -(\gamma^2)^2(\gamma^0)^2 = -(-\mathbb{1})\cdot\mathbb{1} = \underline{\mathbb{1}} \quad \checkmark$$

$$\mathcal{C}^\dagger = (i\gamma^2\gamma^0)^\dagger = -i\gamma^{0\dagger}\gamma^{2\dagger} = -i\gamma^0(-\gamma^2) = -i\gamma^2\gamma^0$$

$$= -\mathcal{C} \quad \checkmark$$

$$\mathcal{C} = i\gamma^2\gamma^0 = i\begin{pmatrix} 0 & +\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} +i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

$$\Rightarrow \mathcal{C}^T = \begin{pmatrix} +i\sigma^{2T} & 0 \\ 0 & -i\sigma^{2T} \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & +i\sigma^2 \end{pmatrix} = -\mathcal{C} \quad \checkmark$$

(c) $\Gamma \equiv \mathbb{1}$, $\mathbb{1}^T = \mathbb{1} \rightarrow \mathcal{C}\mathcal{C}^T = +\mathbb{1}$, $\mathcal{C}\mathbb{1}^T\mathcal{C}^T = +\mathbb{1}$

$$\Gamma = \gamma^\mu \quad \Rightarrow \boxed{\mathcal{C}\gamma^{\mu T}\mathcal{C}^T} = \begin{Bmatrix} \mathcal{C}\gamma^\mu\mathcal{C}^T & \text{for } \mu = 0,2 \\ -\mathcal{C}\gamma^\mu\mathcal{C}^T & \text{for } \mu = 1,3 \end{Bmatrix} =$$

as $\mathcal{C}$ commutes with γ^1, γ^3 and anticommutes with γ^0, γ^2

$$= \boxed{-\gamma^\mu C C^T = -\gamma^\mu}$$

$$\gamma^{5T} = (i\gamma^0\gamma^1\gamma^2\gamma^3)^T = i\gamma^{3T}\gamma^{2T}\gamma^{1T}\gamma^{0T}$$

$$= (-)^2 i\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^0\gamma^3\gamma^2\gamma^1 = -i\gamma^0\gamma^1\gamma^3\gamma^2 = \gamma^5$$

$$\Rightarrow \boxed{C\gamma^{5T}C^T = C\gamma^5 C^T = \gamma^5 CC^T = +\gamma^5}, \text{ as}$$

$$0 = [\gamma^5, i\gamma^2\gamma^0] = [\gamma^5, C]$$

$$\boxed{C(\gamma^\mu\gamma^5)^T C^T} = C\gamma^{5T}\gamma^{\mu T}C^T = C\gamma^{5T}C^T C\gamma^{\mu T}C^T$$

$$= -\gamma^5\gamma^\mu = \boxed{+(\gamma^\mu\gamma^5)}$$

$$\boxed{C\sigma^{\mu\nu T}C^T} = C\frac{i}{2}[\gamma^\mu,\gamma^\nu]^T C^T = -\frac{i}{2}C[\gamma^{\mu T},\gamma^{\nu T}]C^T$$

$$= -\frac{i}{2}[C\gamma^{\mu T}C^T, C\gamma^{\nu T}C^T] = -\frac{i}{2}[\gamma^\mu,\gamma^\nu] = \boxed{-\sigma^{\mu\nu}}$$

(d) $$(\gamma^0)^2 = \begin{pmatrix} (\sigma^2)^2 & 0 \\ 0 & (\sigma^2)^2 \end{pmatrix} = \mathbb{1} \checkmark \quad (\gamma^2)^2 = \begin{pmatrix} -(\sigma^2)^2 & 0 \\ 0 & -(\sigma^2)^2 \end{pmatrix} = -\mathbb{1}$$

$$(\gamma^1)^2 = \begin{pmatrix} -(\sigma^3)^2 & 0 \\ 0 & -(\sigma^3)^2 \end{pmatrix} = -\mathbb{1} \checkmark \quad (\gamma^3)^2 = \begin{pmatrix} -(\sigma^1)^2 & 0 \\ 0 & -(\sigma^1)^2 \end{pmatrix} = -\mathbb{1} \checkmark$$

$$\{\gamma^0, \gamma^{1/3}\} = \begin{pmatrix} 0 & \mp i\{\sigma^2, \sigma^{1/3}\} \\ \mp i\{\sigma^2, \sigma^{1/3}\} & 0 \end{pmatrix} = 0.$$

$$\{\gamma^0, \gamma^2\} = \begin{pmatrix} (\sigma^2)^2 & 0 \\ 0 & -(\sigma^2)^2 \end{pmatrix} - \begin{pmatrix} (\sigma^2)^2 & 0 \\ 0 & -(\sigma^2)^2 \end{pmatrix} = 0.$$

$$\{\gamma^1, \gamma^3\} = i(-i)\begin{pmatrix} \{\sigma^3, \sigma^1\} & 0 \\ 0 & \{\sigma^3, \sigma^1\} \end{pmatrix} = 0.$$

$$\{\gamma^2, \gamma^{1/3}\} = \pm i\begin{pmatrix} 0 & -\{\sigma^2, \sigma^{3/1}\} \\ \{\sigma^2, \sigma^{3/1}\} & 0 \end{pmatrix} = 0. \quad \checkmark$$

$$P_+ + P_- = \frac{1}{2}\left(1+\sigma^2+1-\sigma^2\right) = \mathbb{1} \checkmark \quad , \quad P_\pm^2 = \frac{1}{4}\left(1 \pm \sigma^2\right)^2$$

$$= \frac{1}{2}\left(1 \pm \sigma^2\right) = P_\pm \checkmark \qquad P_+ P_- = \frac{1}{4}\left(1+\sigma^2\right)\left(1-\sigma^2\right) =$$

$$\frac{1}{4}\left(1 - (\sigma^2)^2\right) = 0 . \checkmark ,$$

$$U \gamma^i_{\text{chiral}} U^\dagger = \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix}$$

$$= \begin{pmatrix} -P_+\sigma^i & P_-\sigma^i \\ P_-\sigma^i & P_+\sigma^i \end{pmatrix} \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix}$$

$$= \begin{pmatrix} -P_+\sigma^i P_- + P_-\sigma^i P_+ & -P_+\sigma^i P_+ - P_-\sigma^i P_- \\ P_-\sigma^i P_- + P_+\sigma^i P_+ & P_-\sigma^i P_+ - P_+\sigma^i P_- \end{pmatrix}$$

$$P_\pm \sigma^2 = \pm P_\pm \quad , \quad \sigma^2 P_\pm = \pm P_\pm \implies P_\pm \sigma^2 P_\pm = \pm P_\pm$$

$$P_\mp \sigma^2 P_\pm = 0$$

$$P_\pm \sigma^{1,3} = \sigma^{1,3} P_\mp \implies P_\pm \sigma^{1,3} P_\pm = 0 .$$

$$P_\pm \sigma^{1,3} P_\mp = \sigma^{1,3} P_\mp$$

$$= \begin{cases} i=2 & \begin{pmatrix} 0 & -P_+ + P_- \\ P_+ - P_- & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \\[2ex] i=1,3 & \begin{pmatrix} \sigma^i(P_+ - P_-) & 0 \\ 0 & \sigma^i(P_+ - P_-) \end{pmatrix} = \begin{pmatrix} \sigma^i\sigma^2 & 0 \\ 0 & \sigma^i\sigma^2 \end{pmatrix} = \begin{pmatrix} \pm i\sigma^{3,1} & 0 \\ 0 & \pm i\sigma^{3,1} \end{pmatrix} \end{cases}$$

$$\gamma^0_{\text{Maj.}} = \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix} = \begin{pmatrix} P_+ & P_- \\ -P_- & P_+ \end{pmatrix} \begin{pmatrix} P_- & P_+ \\ P_+ & -P_- \end{pmatrix}$$

$$= \begin{pmatrix} P_+P_- + P_-P_+ & P_+ - P_- \\ P_+ - P_- & -P_+P_- - P_-P_+ \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

$\square$

For the charge conjugation matrix we need:

$\mathcal{C}(\gamma^\mu)^T\mathcal{C}^T = -\gamma^\mu$. We have $(\gamma^0)^T = \begin{pmatrix} 0 & \sigma^{2T} \\ \sigma^{2T} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = -\gamma^0$,

$$(\gamma^1)^T = \begin{pmatrix} i\sigma^{3T} & 0 \\ 0 & i\sigma^{3T} \end{pmatrix} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} = \gamma^1$$

$$(\gamma^2)^T = \begin{pmatrix} 0 & \sigma^{2T} \\ -\sigma^{2T} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = \gamma^2$$

$$(\gamma^3)^T = \begin{pmatrix} 0 & -i\sigma^{1T} \\ -i\sigma^{1T} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} = \gamma^3$$

$$\mathcal{C}(\gamma^\mu)^T\mathcal{C}^T = \begin{Bmatrix} \mathcal{C}\gamma^i\mathcal{C}^T, \; i=1,2,3 \\ -\mathcal{C}\gamma^0\mathcal{C}^T \end{Bmatrix} \overset{!}{=} -\gamma^\mu$$

$\Rightarrow \{\mathcal{C}, \gamma^i\} = 0$ and $[\mathcal{C}, \gamma^0] = 0 \Rightarrow \mathcal{C} \propto \gamma^0$

in order to have $\mathcal{C}^{-1} = -\mathcal{C}$, we need $\boxed{\mathcal{C} = \mp i\gamma^0}$