DESY Theory Group, Hamburg

Exam for the Lecture

Quantum Mechanics II

WS 20/21

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Exam

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Problem 1 Multi-particle wavefunctions

There are three solid state electrons in the three different states $|\psi_\alpha\rangle \equiv |\alpha\rangle$, $|\psi_\beta\rangle \equiv |\beta\rangle$, and $|\psi_\gamma\rangle \equiv |\gamma\rangle$. Write down the properly normalized many-body quantum mechanical wavefunction for the three-electron system.

[**2 points**]

Problem 2 One-particle observables for solid state electrons

Consider the many-body system of solid-state electrons, with two different one-particle observables

$$\hat{\mathcal{O}}_1 = \sum_{i,j=1}^{\infty} O_{ij}^{(1)} \hat{c}_i^\dagger \hat{c}_j \quad , \qquad \hat{\mathcal{O}}_1' = \sum_{i,j=1}^{\infty} O_{ij}^{(1)\prime} \hat{c}_i^\dagger \hat{c}_j \qquad . \tag{1}$$

(a) When do such two observables, $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_1'$ commute with each other, i.e what conditions must the one-particle matrix elements fulfill?

[**4 points**]

(b) Does this hold also for observables of a many-body system of bosons?

[**2 points**]

Problem 3 One-dimensional model of phonons

The task is to quantize the linear chain of N next-neighbor coupled degrees of freedom from exercise sheet 7, repetition exercise 7 (with periodic boundary conditions). This serves as a

one-dimensional model for vibrational phonon excitations in a solid. Consider the Hamiltonian

$$\hat{H}_0 = \sum_{n=1}^{N} \frac{\hat{p}_n^2}{2m} + \sum_{n=1}^{N} \frac{\kappa}{2} \left(\hat{q}_{n+1} - \hat{q}_n\right)^2 \ , \quad [\hat{q}_n, \hat{q}_{n'}] = [\hat{p}_n, \hat{p}_{n'}] = 0, \quad [\hat{q}_n, \hat{p}_{n'}] = i\hbar\delta_{nn'} \tag{2}$$

with the canonical commutators of its N coordinates and momenta $\hat{q}_k, \hat{p}_k, k = 1, \ldots N$.

(a) Introduce the following creation and annihilation operators of one-particle vibrational modes (phonons) to diagonalize the Hamiltonian:

$$\hat{q}_n = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left(z_n^k \hat{b}_k + (z_n^k)^* \hat{b}_k^\dagger \right) , \tag{3a}$$

$$\hat{p}_n = -i \sum_k \sqrt{\frac{\hbar m\omega_k}{2}} \left(z_n^k \hat{b}_k - (z_n^k)^* \hat{b}_k^\dagger \right) \quad . \tag{3b}$$

Here, the z_n^k are the kth powers of the nth N-root of unity (with periodicity in $k \to k+N$),

$$z_n^k = \frac{1}{\sqrt{N}} e^{2\pi i k \frac{n}{N}} \quad . \tag{4}$$

Proof their orthonormality and completeness:

$$\sum_{n=1}^{N} z_n^{k'*} z_n^k = \delta_{kk'} \qquad \sum_{k=1}^{N} z_{n'}^{k\,*} z_n^k = \delta_{nn'} \quad . \tag{5}$$

Invert the relations in Eq. (3) and show that $\hat{b}_k, \hat{b}_{k'}^\dagger$ fulfill canonical commutation relations.

[**5 points**]

(b) Proof

$$\sum_{n=1}^{N} z_n^{k'} z_n^k = \sum_{n=1}^{N} z_n^{k'*} z_n^{k*} = \delta_{k,-k'} \qquad z_{n\pm1}^k = e^{\pm 2\pi i \frac{k}{N}} z_n^k \quad . \tag{6}$$

Diagonalize the Hamiltonian. In order for the off-diagonal terms (two creation or two annihilation operators) to cancel out, derive the condition on the eigenfrequencies ω_k in terms of κ, m.

[**6 points**]

(c) Show that an external force (e.g. from a constant electric field) of the form

$$\hat{H} = \hat{H}_0 + \hat{V} \quad , \qquad \hat{V} = -\sum_n K_n \hat{q}_n \tag{7}$$

does not change the phonon spectrum, but just simultaneously lowers all energies by a certain amount. Calculate that amount in terms of the K_n. *Hint: You need a transformation on the creation and annihilation operators to rediagonalize the Hamiltonian.*

[**4 points**]

Problem 4 Canonical ensemble

For a canonical ensemble with parameter $\beta = 1/T$ (inverse temperature), the probability distribution for energy eigenstates $\{|n\rangle\}$ with energy $\{E_n\}$ is given as

$$W_n = \frac{1}{Z} \exp\left[-\beta E_n\right] \qquad . \tag{8}$$

Here, the partition function Z is independent of n and determined by the condition $1 = \sum_n W_n$. Show the following relations for the expectation value and variance of the energy:

$$-\frac{d}{d\beta} \ln Z = \langle E \rangle \tag{9}$$

$$\frac{d^2}{d\beta^2} \ln Z = \left\langle E^2 \right\rangle - \langle E \rangle^2 \tag{10}$$

[**4 points**]

Problem 5 Canonical ensemble of harmonic oscillators

Calculate the partition function Z for a canonical ensemble of harmonic oscillators with energy spectrum $\left\{E_n = \hbar\omega \left(n + \frac{1}{2}\right) | n = 0, 1, ...\right\}$ by summing the infinite series. Show that the zero point energy drops from the probability distribution, and redefine the partition function. What is the physics behind this redefinition? Determine the expectation values $\langle E \rangle$ and $\langle E^2 \rangle$, and discuss the limits $\beta\omega \to 0$ und $\beta\omega \to +\infty$ for $\langle E \rangle$. What energy distribution do you get, and what do the limits mean physically?

[**6 points**]

Problem 6 Time evolution of Klein-Gordon field operator

Consider the field operator of a Klein-Gordon field for scalar relativistic particles in the Heisenberg picture,

$$\hat{\phi}(x) = \int \widetilde{dp} \left(\hat{a}_p e^{-ip\cdot x} + \hat{a}_p^\dagger e^{+ip\cdot x}\right) \qquad \widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3 (2E_p)} \tag{11}$$

Show that it fulfills the Schrödinger equation,

$$i\frac{\partial}{\partial t}\hat{\phi}(x) = [\hat{\phi}(x), \hat{H}] \qquad , \tag{12}$$

where the Hamiltonian was derived in the lecture as

$$\hat{H} = \int \widetilde{dp}\, E_p\, \hat{a}_p^\dagger \hat{a}_p \qquad , \tag{13}$$

and the canonical commutation relations of the creation and annihilation operators are given by

$$[\hat{a}_p, \hat{a}_q^\dagger] = (2E_p)(2\pi)^3 \delta^3(\vec{p} - \vec{q}) \; , [\hat{a}_p, \hat{a}_q] = 0 \qquad . \tag{14}$$

[**4 points**]

Problem 7 System of two complex scalar fields

The Lagrangian (density) for two complex scalar fields Φ_1, Φ_2 with quartic interaction is given by

$$\mathcal{L} = \sum_{i=1}^{2} \partial_\mu \Phi_i^* \, \partial^\mu \Phi_i - m^2 \sum_{i=1}^{2} \Phi_i^* \Phi_i - \frac{\lambda^2}{2} \left(\sum_{i=1}^{2} \Phi_i^* \Phi_i \right)^2 .$$

(a) Derive the equations of motion. Take Φ and Φ^* as elementary dynamic degrees of freedom (real and imaginary part of Φ is also possible, but this way it is easier!).

[**4 points**]

(b) Show that $\mathcal{L}$ is invariant under the infinitesimal transformation

$$\Phi_i \longrightarrow \Phi_i' = \Phi_i + i\epsilon_a \frac{\sigma^a_{ij}}{2} \, \Phi_j \, ,$$

where σ^a are the usual Pauli matrices

$$\sigma^1 = -\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = -\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma^3 = -\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{15}$$

with commutation relations $[\sigma^a, \sigma^b] = i\epsilon^{abc}\sigma^c$. ϵ^a are constant real parameters. What are the corresponding Noether currents? *Hint:* The formula for the Noether current can be found in Eq. (1) of exercise sheet 10.

[**6 points**]

(c) By using the equations of motions proof that the Noether currents are indeed conserved, $\partial^\mu j^a_\mu = 0$.

[**3 points**]

(d) Calculate the energy-momentum tensor $T_{\mu\nu}$. *Hint:* The formula is in Eq. (14) of exercise sheet 12.

[**4 points**]

Problem 8 Dirac matrices and all that ...

In the lecture, it was shown that the Dirac gamma matrices can be used to construct the spinor representation of the Lorentz algebra. Consider the so-challed chiral representation of the gamma matrices:

$$\gamma^\mu_{\text{chiral}} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \qquad \text{i.e. } \gamma^0_{\text{chiral}} = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} \qquad \gamma^i_{\text{chiral}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad . \tag{16}$$

Again, the σ^i are the Pauli matrices.

(a) Show the behavior of the chiral representation under transposition, $(\gamma^\mu)^T =$?

[**2 points**]

(b) Consider the charge conjugation matrix, $\mathcal{C} = i\gamma^2\gamma^0$. Proof the following properties:

$$\mathcal{C}^{-1} = -\mathcal{C} \qquad \mathcal{C}^\dagger = -\mathcal{C} \, , \qquad \mathcal{C}^T = -\mathcal{C} \, , \tag{17}$$

Calculate its form in the chiral representation.

[**2 points**]

(c) Calculate the following quantities

$$\mathcal{C}\Gamma^T\mathcal{C}^T = \ldots \text{ for } \quad \Gamma = \left\{\mathbb{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\right\} \quad . \tag{18}$$

Here, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

[**4 points**]

(d) From the lecture we know that one switches between particles and antiparticles by complex conjugation in fields, and that a real field corresponds to particles being their own antiparticles. In order to discuss neutral spin 1/2 fermions, it is favorable to use a representation of gamma matrices where the Dirac equation $(i\partial\!\!\!/ - m)\psi(x) = 0$ becomes real. For this, the gamma matrices need to be purely imaginary. Show that this set

$$\gamma^0_{\text{Majorana}} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma^1_{\text{Majorana}} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{19}$$

$$\gamma^2_{\text{Majorana}} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma^3_{\text{Majorana}} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \tag{20}$$

is indeed a representation of the Dirac algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times4}$.

Proof that $\mathcal{P}_\pm = \frac{1}{2}\left(1 \pm \sigma^2\right)$ are indeed projectors, and that the matrix U

$$U = \begin{pmatrix} \mathcal{P}_- & \mathcal{P}_+ \\ \mathcal{P}_+ & -\mathcal{P}_- \end{pmatrix} \quad , \qquad \gamma^\mu_{\text{Majorana}} = U\gamma^\mu_{\text{chiral}}U^\dagger \tag{21}$$

indeed transfers the chiral into the Majorana representation.

In the Majorana representation, in order to define the condition necessary to convert spinors into complex conjugated spinors, the charge conjugation matrix needs to fulfill $u = \mathcal{C}\overline{v}^T$, which implies $\mathcal{C}(\gamma^\mu)^T\mathcal{C}^T$. It also has to fulfill Eq. (17). Which matrix does the job of the charge-conjugation matrix in this representation?

[**6 points**]