DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Let there be light! Today it is all about photons and the electromagnetic field!

Home Exercise 31 Commutators of photon creation and annihilation operators

The electromagnetic field (operator) $\hat{A}_\mu$ has the following Fourier decomposition into one-particle plane wave(-functions)

$$\hat{A}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \Big[\underbrace{e^{-ikx}}_{g_{\vec{k}}(x)} \underbrace{\sum_\lambda \epsilon_\mu^{(\lambda)}(k)\hat{a}^{(\lambda)}(k)}_{\hat{a}_\mu(k)} + \underbrace{e^{ikx}}_{g^*_{\vec{k}}(x)} \underbrace{\sum_\lambda \epsilon_\mu^{(\lambda)*}(k)\hat{a}^{(\lambda)\dagger}(k)}_{\hat{a}^\dagger_\mu(k)} \Big], \tag{1}$$

where the constant 4-vectors are given by

$$\epsilon_\mu^{(i)} = \delta_{\mu i}, \qquad \text{i.e. e.g. } \epsilon_\mu^{(2)} = (0,1,0,0)^T \text{ etc.} \tag{2}$$

Proof the completeness relations

$$\sum_\mu g_{\mu\mu} \epsilon_\mu^{(\lambda)}(\vec{k}) \epsilon^{\mu,(\lambda')*}(\vec{k}) = \delta_{\lambda\lambda'}, \qquad \sum_\lambda g_{\lambda\lambda}\, \epsilon_\mu^{(\lambda)}(\vec{k}) e_\nu^{(\lambda)*}(\vec{k}) = g_{\mu\nu} \tag{3}$$

as well as the equivalence of the following commutators

$$[\hat{a}_\mu(\vec{k}), \hat{a}^\dagger_\nu(\vec{k}')] = -g_{\mu\nu}(2\pi)^3 2\omega \delta^3(\vec{k} - \vec{k}'), \tag{4}$$

and

$$[\hat{a}^{(\lambda)}(\vec{k}), \hat{a}^{(\lambda')\dagger}(\vec{k}')] = -g_{\lambda\lambda'}(2\pi)^3 2\omega \delta^3(\vec{k} - \vec{k}'). \tag{5}$$

Here shall be $g_{\lambda\lambda'}$ component-wise identical to the Minkowski metric, $g_{\mu\nu}$.

Home Exercise 32 Spin 1, Helicity, indefinite Fock space

As photons have spin1, they transform under the adjoint representation of the rotation subgroup of the Lorentz group whose generators are always given by the structure constants. For $SU(2)$ they are given by the totally antisymmetric tensor

$$(T^a_{\text{adj.}})_{bc} = -\mathrm{i}\epsilon_{abc}. \tag{6}$$

(a) Show that these are indeed a representation of the rotation group $SU(2)$. Use the Jacobi identity.

(b) Construct explicitly the spin matrices $\vec{J}$ for spin 1. Determine the normalized eigenvectors $\vec{\epsilon}^{\pm}(\vec{k})$ of the helicity operator $\vec{J} \cdot \vec{k}/|\vec{k}|$ for the special case of a momentum in z direction. (How many eigenvectors there are?)

(c) Show:

$$\vec{\epsilon}^{\pm*}(\vec{k}) \cdot \vec{\epsilon}^{\pm}(\vec{k}) = 1, \quad \vec{\epsilon}^{\pm*}(\vec{k}) \cdot \vec{\epsilon}^{\mp}(\vec{k}) = 0, \quad \vec{k}\vec{\epsilon}^{\pm}(\vec{k}) = 0. \tag{7}$$

(d) Now consider the following four polarization vectors

$$\epsilon_\mu^\pm(\vec{k}) = \begin{pmatrix} 0 \\ \vec{\epsilon}^\pm \end{pmatrix}, \quad \epsilon_\mu^L(\vec{k}) = -\mathrm{i}k^\mu = -\mathrm{i}(|\vec{k}|, \vec{k}), \quad \epsilon_\mu^S(\vec{k}) = \frac{\mathrm{i}}{2|\vec{k}|^2}(|\vec{k}|, -\vec{k}). \tag{8}$$

(The L and S signify longitudinal and scalar mode, respectively.) Determine for the four different (physical and unphysical) polarization vectors, $\sigma, \tau = +, -, L, S$ the metric of the four states corresponding to $\epsilon_\mu^{(\sigma)}(\vec{k})\epsilon^{\mu,(\tau)*}(\vec{k}) = -\tilde{g}^{\sigma\tau}$. Show that it is identical to its inverse, and proof the completeness relation

$$\sum_{\sigma,\tau} \epsilon_\mu^{(\sigma)}(\vec{k})\tilde{g}^{\sigma\tau}\epsilon_\nu^{(\tau)*}(\vec{k}) = -g_{\mu\nu}. \tag{9}$$

(e) In this helicity basis, the photon field operator is given by

$$\hat{A}_\mu(x) = \int \tilde{dk} \sum_{\sigma=\pm,L,S} \left[\hat{a}(\vec{k},\sigma) \underbrace{\epsilon_\mu^{(\sigma)}(\vec{k})e^{-\mathrm{i}kx}}_{f_{\vec{k},\mu}^{(\sigma)}(x)} + \hat{a}^\dagger(\vec{k},\sigma) \underbrace{\epsilon_\mu^{(\sigma)}(\vec{k})e^{-\mathrm{i}kx}}_{f_{\vec{k},\mu}^{(\sigma)*}(x)} \right]. \tag{10}$$

Deduce from the previous exercise the commutators

$$[a(\vec{k},\sigma), a^\dagger(\vec{k}',\sigma)] = (2\pi)^3(2\omega)\tilde{g}^{\sigma\tau}\delta^3(\vec{k}-\vec{k}'). \tag{11}$$

(f) Calculate the norm of the states $(a^\dagger(\vec{k},L) - a^\dagger(\vec{k},S))\,|0\rangle$, $a^\dagger(\vec{k},L)\,|0\rangle$, $a^\dagger(\vec{k},L)\,|0\rangle$. What does this mean for the Fock space of multi-photons states? *Hint:* You might take the square of a delta function simply as a constant.

(g) Show that ($g_{\vec{k}}(x)$ has been defined in the previous exercise):

$$f_{\vec{k},\mu}^{(L)}(x) = \partial_\mu g_{\vec{k}}(x), \qquad \partial^\mu f_{\vec{k},\mu}^{(S)}(x) = \delta^{\sigma,S} g_{\vec{k}}(x).$$

Home Exercise 33 Maxwell equations and energy-momentumm tensor of the electromagnetic field

The classical electromagnetic field without external sources has the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{12}$$

is the field strength tensor.

(a) Derive the Maxwell equations as Euler-Lagrange equations from the Lagrangian above, using $A_\mu(x)$ as the dynamical fields. Transfer the result into the standard form, using $E^i = F^{i0}$ and $\epsilon^{ijk}B^k = -F^{ij}$. How does the action look like with external sources?

(b) What are the components of the dual field strength tensor,

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad . \tag{13}$$

Show that this automatically conserved, $\partial^\mu \tilde{F}_{\mu\nu}$, and that this implies the homogeneous Maxwell equations.

(c) Use the default procedure to construct the energy-momentum tensor from a Lagrangian for a field ϕ

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial_\mu\phi}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{14}$$

for the electromagnetic field. Show that for the general case for a field ϕ it is conserved.

Show that the canonical energy-momentum tensor from the formula above is not symmetric for the electromagnetic field above. One can define a new energy-momentum tensor by

$$T'_{\mu\nu} = T_{\mu\nu} + \partial^\lambda K_{\lambda\mu\nu}, \tag{15}$$

where $K_{\lambda\mu\nu}$ is antisymmetric in its first two indices. Why does this not spoil energy-momentum conservation? Show that the choice

$$K_{\lambda\mu\nu} = F_{\mu\lambda}A_\nu \tag{16}$$

leads to a symmtric energy-momentum tensor, and show that this leads to the default form of the energy density and momentum current density (Poynting vector) of the electromagnetic field:

$$\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right), \qquad \vec{S} = \vec{E} \times \vec{B}. \tag{17}$$