DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 11

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Home Exercise 28 Gordon identity for spinors

Derive the Gordon identity for spinor solutions of the Dirac equation

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left[\frac{p^\mu + p'^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \qquad . \tag{1}$$

Here, q is the momentum transfer, $q = (p' - p)$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. *Hint:* Remember that $u(p)$ is a solution of the Dirac equation, $(\not{p} - m)u(p) = 0$.

Home Exercise 29 Noether currents of the Dirac theory

Show that the transformation of the Dirac spinors $\Psi \to \exp\left[-i\alpha\right]\Psi$ is a symmetry transformation of the Dirac Lagrangian. Use the exercises from the last sheet, i.e. the general formula for the Noether current, to derive the Noether current for this symmetry transformation. Show that this current is conserved. Do you remember what invariance under transformations with complex phases signified in multi-particle quantum mechanics?

What needs to be fulfilled that also $\Psi \to \exp\left[-i\alpha\gamma^5\right]\Psi$ is a symmetry of the Lagrangian?

Home Exercise 30 Projecting out annihilation and creation operators for the Dirac field

Use the Fourier decomposition of the Dirac field operator

$$\hat{\psi}(x) = \int \widetilde{dp} \sum_{s=1}^{2} \left(\hat{b}_p^s u^s(p) e^{-ipx} + \hat{d}_p^{s\dagger} v^s(p) e^{+ipx}\right)$$

as well as of the adjoint field operator $\overline{\psi}(x)$ to project out the creation and annihilation operators for Dirac particles and anti-particles out of them:

$$\hat{b}_k^s = \int d^3x\, \overline{u^s}(k)\, e^{ikx}\, \gamma^0 \hat{\psi}(x) \tag{2a}$$

$$\hat{d}_k^s = \int d^3x\, \hat{\overline{\psi}}(x)\, e^{ikx}\, \gamma^0 v^s(k) \tag{2b}$$

$$\hat{b}_k^{s\dagger} = \int d^3x\, \hat{\overline{\psi}}(x)\, e^{-ikx}\, \gamma^0 u^s(k) \tag{2c}$$

$$\hat{d}_k^{s\dagger} = \int d^3x\, \overline{v^s}(k)\, e^{-ikx}\, \gamma^0 \hat{\psi}(x) \tag{2d}$$

For this, the different normalization relations of spinor solutions of the Dirac equation might be helpful:

$$\overline{u^s}(k)u^{s'}(k) = 2m\delta_{ss'}, \qquad u^{s\dagger}(k)u^{s'}(k) = 2k^0\delta_{ss'}, \tag{3a}$$

$$\overline{v^s}(k)v^{s'}(k) = -2m\delta_{ss'}, \qquad v^{s\dagger}(k)v^{s'}(k) = 2k^0\delta_{ss'}, \tag{3b}$$

$$\overline{u^s}(k)v^{s'}(0) = 0, \qquad v^{s\dagger}(\vec{k})v^{s'}(-\vec{k}) = 0 \tag{3c}$$

Repetition QM I Exercise 13 Magnetic moment of the hydrogen atom in its ground state

The Hamiltonian for a hydrogen atom in a homogeneous magnetic field $\vec{B}$ with $\vec{A}(\hat{\vec{x}}) = \frac{1}{2}\,\vec{B} \times \hat{\vec{x}}$ is given by

$$H = \sum_{i=1}^{2} \left\{ \frac{1}{2m_i} \left(\vec{p}_i - \frac{e_i}{c} \vec{A}(\vec{r}_i) \right)^2 - g_i \vec{S}_i \vec{B} \right\} + \frac{e_1 e_2}{|\vec{r}_1 - \vec{r}_2|} \qquad . \tag{4}$$

Here, the numbers 1 and 2 label the electron and the hydrogen nucleus, respectively.

(a) Use the exercise from sheet 9 to separate $\hat{H}$ with respect to relative and center-of-mass coordinates and derive the operator for the total magnetic moment for the total system. All terms quadratic in the external magnetic field $\mathcal{O}(\vec{B}^2)$ in the Hamiltonian could be neglected. Why are the terms that are not simply relative and total angular momentum in the magnetic moment?

b) The wave function for the ground state shall be given by the following separation ansatz $\Psi_0(\vec{R}, \vec{x}) = \Psi(\vec{R}) e^{-\alpha|\vec{r}|} \chi_p(m) \chi_{e^-}(m')$, where m and m' are the z components of the spins of proton and electron, and χ their corresponding spinor wave functions. Calculate the expectation value of the magnetic moment $\left\langle \hat{\vec{\mu}} \right\rangle$ in the ground state.