DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Prof. Dr. J. Reuter
(Bldg. 2a/304), 3895
P. Bredt/F. Fabry; B. Richard; S. van Thurenhout; T. Wening;

Tutorial 10

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Home Exercise 24 Proof of the Noether theorem

Consider a Lagrangian of a scalar field $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ which is supposed to be invariant under a specific (infinitesimal) symmetry transformation of the scalar field, $\phi(x) \to \phi'(x) = \phi(x) + \delta_\epsilon \phi(x)$. Use the invariance of the Lagrangian and the Euler-Lagrange equations of motion to show that the Noether current of the symmetry transformation

$$j^\mu(x) := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\epsilon \phi(x) \tag{1}$$

is conserved,

$$\partial_\mu j^\mu(x) = 0 \qquad . \tag{2}$$

Show that the space integral over the zero-component of the current (the conserved charge, the generator of the symmetry transformation) is conserved in time,

$$\frac{d}{dt} Q = \frac{d}{dt} \int d^3\vec{x} j^0(x) = 0 \qquad . \tag{3}$$

Hint: Use the current conservation and the fact that together with the fields the current vanishes at spatial infinity.

Home Exercise 25 Dirac representation of the Dirac matrices

In the lecture, the chiral representation of the Dirac matrices

$$\gamma^\mu_{\text{chiral}} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \; i.e. \; \gamma^0_{\text{chiral}} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i_{\text{chiral}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, i = 1, 2, 3 \tag{4}$$

was discussed, which is most suited for ultra-relativistic particles as there helicity and chirality are the same, strictly in the limit $m \to 0$, or $v/c \to 1$. For the non-relativistic limit, e.g. to discuss relativistic effects of bound electrons in atoms, the Dirac representation is better suited:

$$\gamma^0_{\text{Dirac}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i_{\text{Dirac}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, i = 1, 2, 3 \qquad . \tag{5}$$

(a) Show that the gamma matrices in the Dirac representation obey the Dirac algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}$.

(b) Find a unitary transformation between the chiral and Dirac representation. Can you also prove from the fact that it was shown that chiral representation obeys the Dirac algebra, that also the Dirac representation does?

(c) Determine $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the chiral projectors $\frac{1}{2}(1 \pm \gamma^5)$ as well as $\frac{1}{\sqrt{2}}(1 - \gamma^0\gamma^5)$ in the Dirac representation.

(d) Show that γ^0 is Hermitian and γ^i anti-Hermitian. Does this depend on the explicit matrix representation?

(e) Show that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \qquad \gamma^0(S^{\mu\nu})^\dagger\gamma^0 = S^{\mu\nu}, \qquad \text{where } S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad . \tag{6}$$

(f) Show that

$$\gamma^5 = \frac{-i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma, \qquad \text{mit} \quad \epsilon^{0123} = +1 \tag{7}$$

(g) Show

$$\left\{\gamma^\mu, \gamma^5\right\} = 0, \qquad [\gamma^5, S^{\mu\nu}] = 0 \quad . \tag{8}$$

Home Exercise 26 More properties of Gamma matrices (voluntary), but use the results for Home Ex. 27

Disclaimer: Never use an explicit representation for the gamma matrices in this exercise. Traces over gamma matrices are very handy in all kind of calculations of scattering processes of quantum electrodynamics.

(a) Show that a trace over an odd number of gamma matrices vanishes. For this, use $(\gamma^5)^2 = 1$ and $\{\gamma^5, \gamma^\mu\}$. What does this imply for Tr $[\gamma^\mu]$?

(b) Proof Tr $[\gamma^5] = 0$.

(c) Use the Dirac algebra to show

$$\text{Tr}\,[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \qquad \text{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}). \tag{9}$$

(d) Show Tr $[\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_n}] = 0$ for n odd.

(e) Show Tr $[\gamma^\mu\gamma^\nu\gamma^5] = 0$. Which square of a gamma matrix do you use here?

(f) Calculate Tr $[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5]$ (use an index combination and the symmetry).

(g) Proof the following contraction identities:

$$\gamma^\mu\gamma_\mu = 4 \quad \gamma^\mu\gamma^\rho\gamma_\mu = -2\gamma^\rho \quad \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4g^{\rho\sigma} \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{10}$$

Home Exercise 27 Gamma matrices as a basis

(a) Show that the following 16 matrices (S = scalar, V = vector, T = tensor, A = axial vector, P = pseudoscalar)

$$\Gamma^S = \mathbb{1}, \quad \Gamma^V = \gamma^\mu, \quad \Gamma^T = \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \Gamma^A = \gamma^\mu\gamma^5, \quad \Gamma^P = \gamma^5 \tag{11}$$

are a basis for 4×4-Matrizen, i.e. every 4×4-Matrix can be written as a linear combination of them. For that, demand the vanishing of the following linear combination:

$$\sum_{i=S,V,T,A,P} \lambda^i \Gamma^i = \lambda^S \mathbb{1} + \lambda^V_\mu \gamma^\mu + \lambda^T_{\mu\nu} \sigma^{\mu\nu} + \lambda^A_\mu \gamma^\mu \gamma^5 + \lambda^P \gamma^5 = 0. \tag{12}$$

Mulitply successively with the corresponding matrices, and form the trace.

Repetition QM I Exercise 11 Gauge transformation

The state $\psi(x,t)$ be a solution of the Schrödinger equation for a particle in an electromagnetic field with scalar potential ϕ and vector potential $\vec{A}$.

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\vec{x},t) = \hat{H} \psi(\vec{x},t), \qquad \hat{H} = \frac{1}{2m} \left(\hat{\vec{p}} - \frac{e}{c} \vec{A}(\hat{\vec{x}}) \right)^2 + e\phi(\hat{\vec{x}}) \tag{13}$$

Reminder: $\phi'(\vec{x},t) = \phi(\vec{x},t) + \frac{1}{c} \frac{\partial f(\vec{x},t)}{\partial t}$, $\vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) - \vec{\nabla} f(\vec{x},t)$. Show that the gauge-transformed state

$$\psi'(\vec{x},t) = e^{-\frac{\mathrm{i}e}{\hbar c} f(\vec{x},t)} \psi(\vec{x},t) \tag{14}$$

is a solution of a Schrödinger equation for the same electromagnetic fields $\vec{E}$ and $\vec{B}$, i.e. the gauge-transformed Schrödinger equation.

Hint: Verify first: $(\hat{\vec{p}} - \frac{e}{c}\vec{A}') e^{-\frac{\mathrm{i}e}{\hbar c} f(\vec{x},t)} = e^{-\frac{\mathrm{i}e}{\hbar c} f(\vec{x},t)} (\hat{\vec{p}} - \frac{e}{c}\vec{A})$

Repetition QM I Exercise 12 Magnetic moment in a homogeneous field

Consider a spin-$\frac{1}{2}$ particle like an electron of mass m and charge e in a homogeneous magnetic field $\vec{B}$. The Hamiltonian in such a case is given by

$$\hat{H} = \frac{1}{2m} \left(\hat{\vec{p}}^2 - \frac{e}{c} \vec{A}(\hat{\vec{x}}) \right)^2 - g\, \hat{\vec{S}} \cdot \vec{B} \qquad . \tag{15}$$

Determine the magnetic moment of $\hat{\vec{\mu}}$ of the particle. *Hint:* Remember how to write down the vector potential for a constant homogeneous magnetic field, and also remember that the magnetic moment is given by the negative derivative of the Hamiltonian with respect to the magnetic field,

$$\hat{\vec{\mu}} = -\frac{\partial \hat{H}}{\partial \vec{B}} \qquad . \tag{16}$$