DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

WS 20/21

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Tutorial 9

20.01.2021

Home Exercise 20 Relativistically covariant phase-space integration

Show for integrals for functions of 4-momentum k^μ the following integral identity:

$$\int \widetilde{dk} f(k^\mu) := \int \frac{d^3k}{(2\pi)^3(2k^0)} f(k^\mu)\Big|_{k^0=+\sqrt{\vec{k}^2+m^2}} = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2-m^2)\theta(k^0)f(k^\mu). \tag{1}$$

In which cases is this integral Lorentz invariant?

Home Exercise 21 Kinematics: Mandelstam variables

For simple scattering reactions of relativistic elementary particles with two particles in initial and final state (so called $2 \to 2$ processes) with the kinematics

$$p_1 + p_2 \to q_1 + q_2, \quad p_1^2 = m_1^2, p_2^2 = m_2^2, q_1^2 = m_3^2, q_2^2 = m_4^2 \tag{2}$$

it is useful to introduce the relativistically invariant Mandelstam variables:

$$s \equiv (p_1 + p_2)^2 = (q_1 + q_2)^2 \tag{3a}$$

$$t \equiv (q_1 - p_1)^2 = (p_2 - q_2)^2 \tag{3b}$$

$$u \equiv (q_1 - p_2)^2 = (p_1 - q_2)^2 \tag{3c}$$

(a) Proof the relation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{4}$$

(b) We define the function

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \tag{5}$$

Show that the momentum of two relativistic particles in their center-of-mass system can be written as

$$\vec{p}^{\,2}_{\text{CMS}} = \frac{1}{4s}\lambda(s, m_1^2, m_2^2). \tag{6}$$

Home Exercise 22 Vector representation of Lorentz algebra

Show that the generator for Lorentz transformation in the vector representation, $(M^{\mu\nu})^\alpha{}_\beta = i\left(g^{\alpha\mu}\delta^\nu{}_\beta - g^{\alpha\nu}\delta^\mu{}_\beta\right)$ indeed obeys the Lorentz algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right) \qquad .$$

Home Exercise 23 Creation and annihilation operators of a complex scalar field

Take the field operator of a complex scalar field in the form

$$\hat{\phi}(x) = \int \widetilde{dk} \left(\hat{a}_k e^{-ikx} + \hat{b}_k^\dagger e^{ikx} \right), \qquad \widetilde{dk} \equiv \frac{d^3\vec{k}}{(2\pi)^3(2E_k)} \quad , \tag{7}$$

and the commutation relations of the particle and antiparticles creation and annihilation operatos

$$[\hat{a}_k, \hat{a}_p^\dagger] = [\hat{b}_k, \hat{b}_p^\dagger] = (2E_k)(2\pi)^3\delta^3(\vec{k} - \vec{p}) \quad . \tag{8}$$

Show that the creation and annihilation operators can be projected out of the field (and adjoint field) operator in the following way:

$$\hat{a}_k = i \int d^3x \; e^{ikx} \overleftrightarrow{\partial}_t \, \hat{\phi}(x), \qquad \hat{b}_k = -i \int d^3x \; e^{ikx} \overleftrightarrow{\partial}_t \, \hat{\phi}^\dagger(x). \tag{9}$$

We used the following symbol, $\overleftrightarrow{\partial}$, which stands for

$$A\overleftrightarrow{\partial}B = A(\partial B) - (\partial A)B. \tag{10}$$

Repetition QM I Exercise 9 Regularized incarnation of delta distribution

Show that $\lim_{\epsilon\to 0} \delta_\epsilon(x) \equiv \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}, \epsilon > 0$ is a representation of the delta distribution/function. For this determine the function values of δ_ϵ in the limit $\epsilon \to 0$ and verify that – independent of ϵ – always $\int_{-\infty}^{+\infty} dx \; \delta_\epsilon(x) = 1$ holds. The last step is to show $\int_{-\infty}^{+\infty} dx \; (\lim_{\epsilon\to 0} \delta_\epsilon(x)) f(x) = f(0)$. For this, substitute $x = \epsilon\xi$ and assume that you are allowed to exchange limit and integration.

Repetition QM I Exercise 10 Two-particle problem with central potential

Consider two particles with different masses m_1 and m_2, whose interaction only depends on the relative distance of the particles (central force problem). The Hamiltonian of the system be

$$\hat{H} = \frac{\hat{\vec{p}}_1^{\,2}}{2m_1} + \frac{\hat{\vec{p}}_2^{\,2}}{2m_2} + V(\left|\hat{\vec{r}}_1 - \hat{\vec{r}}_2\right|) \quad . \tag{11}$$

As in our setup for multi-particle quantum mechanics, operators that describe observables of one of the particles commute with those of the other particle ("live in a different Hilbert space"),

$$[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2'] = 0. \tag{12}$$

(a) Are the orbital angular momenta of particles 1 and 2, $\hat{\vec{L}}_1$ und $\hat{\vec{L}}_2$ constants of motion? (i.e. that these operators commute componentwise with the Hamiltonian.) If not, does a linear combination exist which is a constant of motion?

(b) Now transfer the problem to an effective one-particle problem by using a separation ansatz. For this split the coordinates into those for the center-of-mass system and relative coordinates, and show that the total angular momentum of the system $\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$ can be written as

$$\hbar\hat{\vec{L}} = \hat{\vec{R}} \times \hat{\vec{P}} + \hat{\vec{r}} \times \hat{\vec{p}}, \tag{13}$$

where $\hat{\vec{R}} = \frac{m_1\hat{\vec{r}}_1 + m_2\hat{\vec{r}}_2}{m_1+m_2}$, $\hat{\vec{p}} = \frac{m_2\hat{\vec{p}}_1 - m_1\hat{\vec{p}}_2}{m_1+m_2}$, $\hat{\vec{r}} = \hat{\vec{r}}_1 - \hat{\vec{r}}_2$ und $\hat{\vec{P}} = \hat{\vec{p}}_1 + \hat{\vec{p}}_2$. Show that corresponding quantities are canonically conjugated.

(c) We now add another term to Hamiltonian, a so-called spin-orbit coupling term, which contains the total spin of the system, $\hat{\vec{S}}$, which fulfills the relations $[\hat{S}_i, \hat{S}_j] = \mathrm{i}\epsilon_{ijk}\hat{S}_k$, and $[\hat{S}_i, \text{remainder}] = 0$. This adds to the Hamiltonian for the relative motion, $\hat{H}_{\text{rel.}}$ a term $\hat{H}_{\text{SL}} = f(|\hat{\vec{r}}|)(\hat{\vec{r}} \times \hat{\vec{p}}) \cdot \hat{\vec{S}}$. Is $\hat{\vec{L}}$ still a constant of motion, component-wise? If not, find a new constant of motion.