DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 8

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Home Exercise 18 Connection between Lorentz group and $SL(2,\mathbb{C})$

(a) In the lecture the Lorentz transformation matrices for the two smallest non-trivial representations of the Lorentz group, the left-chiral and right-chiral spinor transformation, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, were derived as:

$$\Lambda_L = \Lambda^{(\frac{1}{2},0)} = \exp\left[-\tfrac{i}{2}(\vec{\phi}-i\vec{\nu})\sigma\right] \quad , \tag{1}$$

$$\Lambda_R = \Lambda^{(0,\frac{1}{2})} = \exp\left[-\tfrac{i}{2}(\vec{\phi}+i\vec{\nu})\sigma\right] \quad . \tag{2}$$

Here, $\sigma^i, i = 1,2,3$ are the three Pauli matrices, $\phi^i \in \mathbb{R}$ are the three Euler angles parameterizing rotations, and $\nu^i \in \mathbb{R}$ are the boost parameters for the boosts along the three different axes.

Show that these matrices are not unitary. Furthermore, show that they have unit determinant. Hence, they constitute the group of special linear transformation in two (complex) dimensions, $SL(2,\mathbb{C})$.

(b) For $SL(2,\mathbb{C})$, we introduce the following generalization of the Pauli matrices:

$$\sigma^\mu = (\mathbb{1},\vec{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{1},-\vec{\sigma}) \qquad , \mathbb{1} \equiv \mathbb{1}_{2\times 2} \tag{3}$$

Show that they have the following properties:

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}\cdot\mathbb{1} \tag{4}$$

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}\cdot\mathbb{1} \tag{5}$$

$$\mathrm{Tr}\left[\sigma^\mu\bar{\sigma}^\nu\right] = 2g^{\mu\nu} \tag{6}$$

$$\sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu\,T} \tag{7}$$

$$\sigma^2\bar{\sigma}^\mu\sigma^2 = \sigma^{\mu T} \tag{8}$$

(c) Now, we write down the following representation of a Minkowski 4-vector as a 2×2 matrix:

$$X := x_\mu\sigma^\mu \quad . \tag{9}$$

Write the explicit matrix form of X, show that it is Hermitian, and calculate its determinant. What can you see about the Lorentz transformation properties of the result for the determinant?

(d) Use the results from part (b) to project out the individual x^μ components from X.

(e) Repeat (c) and (d) for the second possibility to map a Minkowski 4-vector to a Hermitian 2×2 matrix,

$$\bar{X} := x_\mu\bar{\sigma}^\mu \quad . \tag{10}$$

(f) Next, we do a $SL(2,\mathbb{C})$ transformation on X,

$$X \mapsto X' = \Lambda_L\, X\, \Lambda_L^\dagger \qquad . \tag{11}$$

What holds for the determinant of X'? Hence, what kind of transformation is $\Lambda_L\, X\, \Lambda_L^\dagger$? Why do we have to take $\Lambda_L^\dagger$ and not Λ_L^{-1} or Λ_L^T?

(g) Use part (d) to derive the form of the vectorial Lorentz transformation $\Lambda^\mu_{\ \nu} x^\nu$ in terms of the so called spin tensors σ^μ and $\bar{\sigma}^\mu$.

(h) From all this, derive now the Lorentz transformation properties of the spin tensors, $\sigma^\mu \mapsto \sigma'^\mu$, taking the transformation of $\bar{X}'$ as

$$\bar{X} \mapsto \bar{X}' = \Lambda_R\, \bar{X}\, \Lambda_R^\dagger \qquad . \tag{12}$$

(i) Convince yourself either from the transformation law Eq. (11) or the transformation law in (h) that indeed the $\phi^i, i = 1,2,3$ parameterize rotations around the i axis, while the ν^i parameterize boosts along the i axis.

Home Exercise 19 Heavyside step function and Green's function

(a) Show that the Heaviside step function is given by the following integral representation:

$$\theta(t)\cdot e^{-iEt} = \int \frac{d\Omega}{2\pi} e^{-i\Omega t} \frac{i}{\Omega - E + i\epsilon} \ . \tag{13}$$

(b) The Green's function (propagator) $G(\vec{x},t;\vec{x}',t')$ for the wave function of a free particle is given by

$$i\Phi(\vec{x},t)\theta(t-t') = \int d^3\vec{x}' G(\vec{x},t;\vec{x}',t')\Phi(\vec{x}',t') \qquad . \tag{14}$$

Find a representation of the propagator in terms of an expansion in a complete orthonormal basis of solutions of the free Schrödinger equation.

Repetition QM I Exercise 7 Clebsch-Gordan coefficients and normalization

When adding two angular momenta to a total angular momentum, $J = j_1 + j_2$, this constitutes just a change of basis in Hilbert space,

$$|J,M\rangle = \sum_{m_1,m_2} C(j_1,m_1;j_2,m_2|\,J,M)\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle \qquad . \tag{15}$$

Determine the Clebsch-Gordan coefficients, which are the coefficients of the expansion of the total angular momentum with respect to the added angular momenta, and proof their normalization

$$\sum_{m_1,m_2} |C(j_1,m_1;j_2,m_2|\,J,M)|^2 = 1 \qquad . \tag{16}$$

Repetition QM I Exercise 8 Coupling of orbital angular momentum and spin

(a) Consider an orthonormal basis $\{|\ell, m_\ell; s, m_s\rangle\} := \{|\ell, m_\ell\rangle \otimes |s, m_s\rangle\}$ of simultaneous eigenkets of orbital angular momentum, $\hat{\vec{L}}^2, \hat{L}_z$ and spin $\hat{\vec{S}}^2, \hat{S}_z$ with

$$\hat{\vec{L}}^2 \,|\ell, m_\ell; s, m_s\rangle = \ell(\ell+1)\,|\ell, m_\ell; s, m_s\rangle \tag{17}$$

$$\hat{\vec{S}}^2 \,|\ell, m_\ell; s, m_s\rangle = s(s+1)\,|\ell, m_\ell; s, m_s\rangle \tag{18}$$

$$\hat{\vec{L}}_z \,|\ell, m_\ell; s, m_s\rangle = m_\ell\,|\ell, m_\ell; s, m_s\rangle \tag{19}$$

$$\hat{\vec{S}}_z \,|\ell, m_\ell; s, m_s\rangle = m_s\,|\ell, m_\ell; s, m_s\rangle \tag{20}$$

with $\ell = 1$ and $s = \frac{1}{2}$.

(a) Couple orbital angular momentum and spin to total angular momentum, $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ with $j = \frac{3}{2}$. Determine an orthonormal basis, $\{|j, m_j\rangle\}$ of simultaneous eigenkets of $\hat{\vec{L}}^2, \hat{\vec{S}}^2, \hat{\vec{J}}^2$, and $\hat{J}_z$ of total angular momentum, orbital angular momentum and spin, but only the z component of the total angular momentum (but not the inidividual m_ℓ, m_s). *Hint:* Determine the uniquely given "highest weight" state $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ and construct all others by repeated application of the lowering operator $\hat{J}_-$. Use the results from Repetition QM I Exercise 5.

(b) Now for general ℓ and $j = \ell + \frac{1}{2}$, show by induction with respect to m_j that

$$\begin{aligned}\left|j = \ell + \frac{1}{2}, m_j\right\rangle = \;& \sqrt{\frac{\ell + m_j + \frac{1}{2}}{2\ell+1}}\,\left|\ell, m_\ell = m_j - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle \\ &+ \sqrt{\frac{\ell - m_j + \frac{1}{2}}{2\ell+1}}\,\left|\ell, m_\ell = m_j + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle \qquad .\end{aligned} \tag{21}$$