DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 7

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Home Exercise 15 Casimir Operators of Poincaré algebra

Casimir operators are by definition those that commute with the whole set of operators (the "underlying algebra") and with each other, so they constitute the maximal set of simultaneously diagonalizable operators. Hence, all states can be expressed by their eigenvalues ("quantum numbers").

The Poincaré algebra of space-time symmetries consists of the boost and rotation generators, $M^{\mu\nu}$, as well as the energy-momentum 4-vector, P^μ, being the generator of time-space translations:

$$[P^\mu, P^\nu] = 0 \tag{1a}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{1b}$$
$$[P^\mu, M^{\rho\sigma}] = i\left(g^{\mu\rho}P^\sigma - g^{\nu\sigma}P^\rho\right) \tag{1c}$$

(The mathematicians would call this the semi-direct product of the rotation group on a space with Minkowski signature and the 4-dim. [Abelian] translation group, $\mathcal{P} \cong SO(1,3) \rtimes \mathbb{R}^4$.)

Proof that the energy-momentum squared, P^2 as well as the square of Pauli-Ljubarski vector, W^2 are Casimir operators of the Poincaré group, where W^μ is defined as

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \qquad , \tag{2}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in four dimension with $\epsilon^{0123} = +1$.

For the proof that W^2 is a Casimir operator, first show that it transforms as a 4-vector under boosts and rotations, i.e.

$$[W_\mu, M^{\rho\sigma}] = i\left(g^\rho_\mu W^\sigma - g^\sigma_\mu W^\rho\right) \qquad . \tag{3}$$

For this use the identity

$$g_{\sigma\lambda}\epsilon_{\mu\nu\alpha\beta} + g_{\sigma\mu}\epsilon_{\nu\alpha\beta\lambda} + g_{\sigma\nu}\epsilon_{\alpha\beta\lambda\mu} + g_{\sigma\alpha}\epsilon_{\beta\lambda\mu\nu} + g_{\sigma\beta}\epsilon_{\lambda\mu\nu\alpha} = 0 \tag{4}$$

Why does it hold?

What is the commutator $[W_\mu, W_\nu]$?

Home Exercise 16 Connection between the Lorentz group $SO(1,3)$ and $SU(2)\times SU(2)$

We take the Lorentz generators $M^{\mu\nu}$ fulfilling the Lorentz algebra, Eq. (1b), from which we define as in the lecture the new generators:

$$J^i := \frac{1}{2}\epsilon^{ijk}M^{jk}, \qquad K^i := M^{0i} \qquad i,j,k = 1,2,3 \qquad . \tag{5}$$

(a) Show that the J^i are Hermitian operators, $(J^i)^\dagger = J^i$, and the K^i are anti-Hermitian operators, $(K^i)^\dagger = -K^i$.

(b) Show that these operators fulfill the commutation relations

$$[J^i, J^j] = i\epsilon^{ijk}J^k, \qquad [J^i, K^j] = i\epsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k, \tag{6}$$

(c) Proof that the two linear combinations from these operators,

$$T_+^i = \frac{1}{2}\left(J^i + iK^i\right) \ , \qquad T_-^i = \frac{1}{2}\left(J^i - iK^i\right) \tag{7}$$

constitute two mutually commuting $SU(2)$ algebras, i.e.

$$[T_\pm^i, T_\pm^j] = i\epsilon^{ijk}T_\pm^k, \qquad [T_+^i, T_-^j] = 0 \qquad . \tag{8}$$

Home Exercise 17 Clifford algebra as representation of Lorentz algebra

The Clifford algebra is an algebra of (in 4 space-time dimensions) 4 anticommuting generators (matrices in an explicit representation), $\Gamma^\mu, \mu = 0,1,2,3$ that are the generators of Lorentz transformations for the spinor or spin-1/2 representation of the Lorentz group. They fulfill the Clifford algebra (which for this special case is also called Dirac algebra):

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}\cdot \mathbb{1}_{4\times 4} \qquad . \tag{9}$$

Show that the generators

$$\Sigma^{\mu\nu} := \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{10}$$

constitute a representation of the Lorentz group (namely for spin 1/2), i.e. they fulfill

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] \ = \ -i\left(g^{\mu\rho}\Sigma^{\nu\sigma} - g^{\mu\sigma}\Sigma^{\nu\rho} - g^{\nu\rho}\Sigma^{\mu\sigma} + g^{\nu\sigma}\Sigma^{\mu\rho}\right) \qquad . \tag{11}$$

Repetition QM I Exercise 4 Eigenvalues of projectors [easy]

The operator $\hat{P}$ has the projector property, $\hat{P}^2 = \hat{P}$. Show that all its eigenvalues are 0 and 1.

Repetition QM I Exercise 5 Angular momentum and matrix representation

We again consider the algebra of angular momentum operators:

$$\left[\hat{J}_i, \hat{J}_j\right] = \mathrm{i}\epsilon_{ijk}\hat{J}_k \tag{12}$$

From the Quantum Mechanics lecture it is known that there are common eigenstates to the square of angular momentum, $\hat{\vec{J}}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$ and one of its components (usually $\hat{J}_3$), denoted by their eigenvalues as $|j, m\rangle$, fulfilling the following eigenvalue equations:

$$\hat{\vec{J}}^2 \left|j, m\right\rangle = j(j+1)\left|j, m\right\rangle \tag{13}$$

$$\hat{J}_3 \left|j, m\right\rangle = m\left|j, m\right\rangle \tag{14}$$

Here, j takes on the following values, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ while m for each *fixed* j is from the set $m = -j, -j+1, -j+2, \dots, j-1, j$. The so-called ladder operators are given by

$$\hat{J}_+ = \hat{J}_1 + \mathrm{i}\hat{J}_2, \qquad \hat{J}_- = \hat{J}_1 - \mathrm{i}\hat{J}_2. \tag{15}$$

(a) Using Eq. (12), calculate the commutators

$$\left[\hat{J}_3, \hat{J}_+\right], \left[\hat{J}_3, \hat{J}_-\right] \text{ and } \left[\hat{J}_+, \hat{J}_-\right]. \tag{16}$$

Convince yourself that $\hat{J}_\pm$ commutes with $\hat{\vec{J}}^2$.

(b) Under the assumption that is not the state "of highest weight", $m \neq j$, show that – if $|j, m\rangle$ is an eigenstate of $\hat{J}_3$ – so is $\left(\hat{J}_+ \left|j, m\right\rangle\right)$ eigenstate to $\hat{J}_3$ as well. Similarly, under the assumption $m \neq -j$, show that also $\left(\hat{J}_- \left|j, m\right\rangle\right)$ is an eigenstate. What are their eigenvalues?

(c) Determine the proportionality constant between $\hat{J}_\pm \left|j, m\right\rangle$ and the corresponding properly normalized eigentstate of $\hat{J}_3$. *Hint:* Express the square of the ladder operators by diagonal operators and assume any appearing phases to be trival.

(d) Now we specialize to a fixed system of angular momentum $j = 1$. Hence, the vectors $\left|1, 1\right\rangle, \left|1, 0\right\rangle, \left|1, -1\right\rangle$ constitute a basis of the representation space. Determine the matrix representations of the operators $\hat{\mathcal{O}} := \hat{J}_+, \hat{J}_-, \hat{J}_3, \hat{J}_2, \hat{J}_1$;

if you like also for $\hat{\vec{J}}^2$. Note that the matrix representation in the basis above looks like:

$$\left(\hat{\mathcal{O}}\right)_{m,m'} = \begin{pmatrix} \left\langle 1,1\middle|\hat{\mathcal{O}}\middle|1,1\right\rangle & \left\langle 1,1\middle|\hat{\mathcal{O}}\middle|1,0\right\rangle & \left\langle 1,1\middle|\hat{\mathcal{O}}\middle|1,-1\right\rangle \\ \left\langle 1,0\middle|\hat{\mathcal{O}}\middle|1,1\right\rangle & \left\langle 1,0\middle|\hat{\mathcal{O}}\middle|1,0\right\rangle & \left\langle 1,0\middle|\hat{\mathcal{O}}\middle|1,-1\right\rangle \\ \left\langle 1,-1\middle|\hat{\mathcal{O}}\middle|1,1\right\rangle & \left\langle 1,-1\middle|\hat{\mathcal{O}}\middle|1,0\right\rangle & \left\langle 1,-1\middle|\hat{\mathcal{O}}\middle|1,-1\right\rangle \end{pmatrix} . \tag{17}$$

(e) By using ordinary matrix multiplication, confirm that the matrices just derived obey the canonical commutation relations for angular momentum, Eq. (12). If you like, also validate Eq. (16).

Repetition QM I Exercise 6 Linear chain

Consider a linear chain, i.e. the quantum mechanical equivalent of n point masses connected by harmonic "springs" (with spring constant D):'

Use periodic boundary conditions, i.e. the nth point mass is interacting again by "spring" with the first point mass.

(a) Write down the Hamiltonian of the system.

(b) Express the potential energy as a quadratic form.

(c) Diagonalize this quadratic form. Use an ansatz of complex vectors for the eigenstates motivated by the periodicity of the system:

$$Q_i = (1, z_i, z_i^2, \ldots, z_i^{n-1}), \qquad z_i^n = 1 \qquad . \tag{18}$$

(d) Interpret the eigenvalues and eigenstates of the system.