DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 6

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Home Exercise 14 Algebra of angular momentum operators

In order to prepare the Lorentz group as a relativistic generalization of the non-relativistic rotational group, we study (or repeat, hopefully) the algebra of angular momentum operators.

(a) As a warm-up, remember that in quantum mechanics every vectorial (actually axial vectorial) operator $\hat{\mathcal{O}}_i, i = 1, 2, 3$ that fulfills the algebra

$$[\hat{\mathcal{O}}_i, \hat{\mathcal{O}}_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} \hat{\mathcal{O}}_k \tag{1}$$

constitutes an angular momentum (more exactly, constitutes a representation of the 3-dim. rotation group $SO(3)$).

Show that the Pauli matrices

$$J_i = \frac{\hbar}{2}\sigma_i, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

fulfill that algebra.

(b) Show that from the Jacobi identity (repetion QM I exercise 3 (i) the following identity for the ε_{ijk} tensor follows:

$$\varepsilon_{ajk}\varepsilon_{ali} + \varepsilon_{ajl}\varepsilon_{aik} + \varepsilon_{aji}\varepsilon_{akl} = 0 \qquad . \tag{3}$$

(c) Using canonical commutation relations, $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ for $i, j = 1, 2, 3$, proof that the orbital angular momentumis an angular momentum as well:

$$\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}} \qquad \Longrightarrow \qquad [\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k \qquad . \tag{4}$$

(d) From now on, we use Einstein summation convention, i.e. repeated indices are summed over, and use units where $\hbar \equiv 1$. Next, we define an antisymmetric tensor out of the components of a general angular momentum (axial) vector operator $\hat{J}_i$ as

$$\hat{J}_{ij} := i\varepsilon_{ijk}\hat{J}_k \qquad . \tag{5}$$

Show that this tensor fulfills the following algebra:

$$[\hat{J}_{ij}, \hat{J}_{kl}] = i\left(\delta_{ik}\hat{J}_{jl} - \delta_{il}\hat{J}_{jk} - \delta_{jk}\hat{J}_{il} + \delta_{jl}\hat{J}_{ik}\right) \qquad . \tag{6}$$

Hint: For this use the identity in Eq. (3), and proof the identity (summation convention!)

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj} \qquad . \tag{7}$$

This can be done by either noting that both sides have the same combinations of antisymmetric indices and symmetric pairs of indices and agree for one combination (and cyclic), or use the connection between the Levi-Cività tensor ε_{ijk} and determinants.

(e) Show that the three antisymmetric matrices (i, j label the matrices, a, b are the row/column indices!)

$$(M^{ij})_{ab} = -i\left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right) \tag{8}$$

form a representation of the algebra of the group of 3-dimensional rotations, Eq. (6).

Write down the explicit form of these three matrices, $(J^1)_{ab} \equiv (M^{23})_{ab}, (J^2)_{ab} \equiv (M^{31})_{ab}, (J^3)_{ab} \equiv (M^{12})_{ab}$.

Proof that $(J^i)_{ab} = -i\epsilon_{iab}$.

Show that they fulfill the angular momentum algebra. Can you avoid a brute force calculation by using Eq. (3)?

Repetition QM I Exercise 2 System of coupled oscillators

Consider a quantum mechanical system with fundamental observables $\hat{q}_a, \hat{p}_a (a = 1, \ldots, N)$, with canonical commutation relations $[\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0$, $[\hat{q}_a, \hat{p}_b] = \mathrm{i}\hbar\delta_{ab}$, and quadratic, but non-diagonal, Hamiltonian:

$$\hat{H} = \sum_{a=1}^{N} \frac{\hat{p}_a^2}{2m} + \frac{1}{2} \sum_{1 \le a,b \le N} V_{ab}\hat{q}_a\hat{q}_b, \tag{9}$$

The coefficents $V_{ab} = V_{ba} \in \mathbb{R}$ form a real symmetric matrix, whose spectrum is positive semi-definite (i.e. all its eigenvalues are non-negative).

Use an orthonormal system of eigenvectors of this matrix with real components, $\left\{U_a^{(s)} | s = 1, \ldots, N\right\}$,

$$U_a^{(s)} = \left(U_a^{(s)}\right)^*, \qquad \sum_{a=1}^{N} U_a^{(s)} U_a^{(s')} = \delta_{ss'}, \qquad \sum_{b=1}^{N} V_{ab} U_b^{(s)} = k_s U_a^{(s)} \tag{10}$$

and define new position and momentum operators $\hat{Q}_s = \sum_a U_a^{(s)}\hat{q}_a$, $\hat{P}_s = \sum_a U_a^{(s)}\hat{p}_a$. Write down the completeness condition of the eigenvector system $\left\{U_a^{(s)} | s = 1, \ldots, N\right\}$.

Show that these operators $\hat{Q}_s, \hat{P}_s$ are Hermitian $\hat{Q}_s = \hat{Q}_s^\dagger$, $\hat{P}_s = \hat{P}_s^\dagger$, obey canonical commutation relations, $[\hat{Q}_s, \hat{P}_{s'}] = \mathrm{i}\hbar\delta_{ss'}$, $[\hat{Q}_s, \hat{Q}_{s'}] = [\hat{P}_s, \hat{P}_{s'}] = 0$, and that the Hamiltonian is given by

$$\hat{H} = \sum_{s=1}^{N} \left(\frac{\hat{P}_s^2}{2m} + \frac{m\omega_s^2}{2}\hat{Q}_s^2 \right) \tag{11}$$

with $m\omega_s^2 = k_s$.

Determine the spectrum of $\hat{H}$.

Repetition QM I Exercise 3 Gymnastics with operators and matrices [easy]

Be $\{|a_k\rangle\}, k = 1, 2, \ldots$ a CONB (complete orthonormal basis). Show the following properties by using the completeness and the definition of Hermitian adjoint operators in terms of their matrix elements, $\left\langle a_k \middle| \hat{\mathcal{O}} \middle| a_l \right\rangle = \left\langle a_l \middle| \hat{\mathcal{O}}^\dagger \middle| a_k \right\rangle^*$:

(a) The trace $\text{Tr}\left[\hat{A}\right]$ of an operator $\hat{A}$ in matrix representation is independent of the choice of basis $\{|a_k\rangle\}$.

(b) $\text{Tr}\left[\hat{A}\hat{B}\right] = \text{Tr}\left[\hat{B}\hat{A}\right]$.

(c) $\left(\hat{A} + \hat{B}\right)^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$.

(d) $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

(e) $\left(\lambda\hat{A}\right)^\dagger = \lambda^*\hat{A}^\dagger$ for $\lambda \in \mathbb{C}$.

(f) The commutator $[\hat{A}, \hat{B}]$ is invariant under a unitary transformation $\hat{U}^\dagger = \hat{U}^{-1}$.

Be $\hat{A}, \hat{B}, \hat{C}$ linear operator; verify the following relations:

(g) $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$.

(h) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(i) $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]]$ (Jacobi identity).