DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 3

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Home Exercise 7 Normalization of N-particle states, determinants, permanents

In the lecture, Slater determinants and permanents

$$|\Psi\rangle = C_s \hat{S}_s |\psi_1, \ldots, \psi_N\rangle = C_s \frac{1}{N!} \sum_{P \in S_N} (-s)^{\chi_P} |\psi_{P(1)}\rangle^{(1)} |\psi_{P(2)}\rangle^{(2)} \ldots |\psi_{P(N)}\rangle^{(N)} \quad , \tag{1}$$

where the sum over the permutations runs over all elements of the symmetric group S_N. For both determinants and permanents we work without loss of generality with orthonormal one-particle states,

$$\langle \psi_i | \psi_j \rangle = \delta_{ij} \quad . \tag{2}$$

From demanding that the N-particle state is normalized to 1,

$$\langle \Psi | \Psi \rangle \overset{!}{=} 1 \quad , \tag{3}$$

show that the normalization constant for fermionic Slater determinants has to be

$$C_{s=1} = \sqrt{N!} \quad . \tag{4}$$

For bosons, use

$$|\Psi\rangle = C_{s=-1} \hat{S}_s \Big| \underbrace{\psi_1, \ldots, \psi_1}_{n_1}, \underbrace{\psi_2, \ldots, \psi_2}_{n_2}, \ldots, \underbrace{\psi_q, \ldots, \psi_q}_{n_q} \Big\rangle, \qquad \sum_{\alpha=1}^{q} n_a = N \quad . \tag{5}$$

to show

$$C_{s=-1} = \sqrt{\frac{N!}{\prod_{a=1}^{q} n_\alpha!}} \quad . \tag{6}$$

Home Exercise 8 Unitary transformation of one-particle states

Proof that changing the one-particle basis by a unitary transformation, $UU^\dagger = \mathbb{1}$ or $\sum_k U_{ik} U^*_{jk} = \delta_{ij}$ and $\det[U] = 1$ on every one-particle Hilbert space,

$$|\psi_k\rangle^{(i)} = \sum_{l=1}^{N} U_{kl} |\psi_l\rangle^{(i)} \tag{7}$$

does not change the Slater determinant.

Home Exercise 9 Orthogonality of Slater determinants and permanents

Use a similar calculation as in Ex. 7 to show under which conditions Slater determinants or permanents are orthogonal to each other. You can formalize this by labelling one-particle states by the ordered energy eigenvalues of a (one-particle) Hamiltonian, $j_1 < j_2 < \ldots < j_N$, as $\epsilon_{j_1} < \ldots < \epsilon_{j_N}$.

Home Exercise 10 Phase transformation and particle number conservation

A Hamiltonian is particle-number conserving, if it commutes with the particle-number operator, $\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}$. This is analogous as doing a global phase transformation of the annihilation and creation operators, and state that the Hamiltonian is invariant under this phase transformation. We take the so-called Bogolyubov Hamiltonian,

$$\hat{H} = \sum_{k>0} \varepsilon_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-k} \right) + \sum_{k>0} V_k \left(\hat{a}_k^\dagger \hat{a}_{-k}^\dagger + \hat{a}_k \hat{a}_{-k} \right) . \tag{8}$$

where $\hat{a}_k, \hat{a}_k^\dagger$ obey canonical commutation relations.

(a) Use the result of previous exercises to show

$$e^{i\alpha\hat{N}}\ \hat{a}_k\ e^{-i\alpha\hat{N}} = \hat{a}_k\ e^{-i\alpha}, \qquad \alpha \in \mathbb{R}\ , \tag{9}$$

and calculate the phase transformation of the whole Hamiltonian:

$$e^{i\alpha\hat{N}}\ \hat{H}\ e^{-i\alpha\hat{N}} = ? \qquad . \tag{10}$$

For which phase α is the Hamiltonian invariant under these phase transformations. Interpret the result physically.

(b) We now consider a local phase transformation, i.e. one that is individual for each single momentum mode,

$$e^{i\sum_k \alpha_k \hat{n}_k}\ \hat{H}\ e^{-i\sum_k \alpha_k \hat{n}_k} \qquad .$$

Are there any local phases α_k, for which the Hamiltonian is invariant? If yes, interpret your result physically.

(c) Replace the bosonic creation and annihilation operators $\hat{a}_k$ by their fermionic counterparts, $\hat{c}_k$, obeying canonical anticommutation relations. Proof that all statements made before for bosons also hold for fermions by showing

$$e^{i\alpha\hat{N}}\ \hat{c}_k\ e^{-i\alpha\hat{N}} = \hat{c}_k\ e^{-i\alpha}, \qquad \alpha \in \mathbb{R}\ , \tag{11}$$