DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Prof. Dr. J. Reuter
(Bldg. 2a/304), 3895
P. Bredt/F. Fabry; B. Richard; S. van Thurenhout; T. Wening;

Tutorial 2

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Home Exercise 4 Matrix elements of Two-Particle Observables

In the lecture, we discussed two-particle observables, with a prime example being two-particle central forces of the N-particle system, e.g. the inter-electron Coulomb interactions in the solid state:

$$\hat{\mathcal{O}}_2 = \sum_{i<j=1}^{N} \mathcal{O}^{(2)}(\hat{\vec{r}}_i, \hat{\vec{p}}_i, \hat{s}_{z,i}; \hat{\vec{r}}_j, \hat{\vec{p}}_j, \hat{s}_{z,j}; t) \qquad . \qquad , \tag{1}$$

Use the spin-orbit representation to show that a general matrix element of such a two-particle observable can be written as

$$\left\langle \Phi \middle| \hat{\mathcal{O}}_2 \middle| \Psi \right\rangle = \frac{1}{2} N(N-1) \left\langle \Phi \middle| \mathcal{O}^{(2)}(\hat{\vec{r}}_1, \hat{\vec{p}}_1, \hat{s}_{z,1}; \hat{\vec{r}}_2, \hat{\vec{p}}_2, \hat{s}_{z,2}; t) \middle| \Psi \right\rangle \qquad , \tag{2}$$

where $|\Phi\rangle, |\Psi\rangle$ are arbitrary states from the totally symmetric or antisymmetric sector of N-particle Hilbert space, and in the last equation the operator acts only on the one-particle Hilbert spaces with labels (1) and (2).

Home Exercise 5 Baker-Campbell-Hausdorff Formula

(a) Prove the Baker-Campbell-Hausdorff formula for a linear operator on Hilbert space,

$$e^A B e^{-A} = \sum_{k=0}^{\infty} \frac{1}{k!} [A, B]_k \qquad , \tag{3}$$

where $[A, B]_0 = B$ and $[A, B]_k = [A, [A, B]]_{k-1}$.

Hint: Replace the operator A by αA with $\alpha \in \mathbb{R}$, and do a Taylor expansion in α.

(b) For the case that $[A, [A, B] = [B, [A, B] = 0$ (the Heisenberg algebra or the creation and annihilation operators of the harmonic oscillator are examples), take the series expansion of e^B to show with the result from (a) that

$$e^A e^B = e^B e^A e^{[A,B]} \qquad . \tag{4}$$

(c) Show that, using again (a), that for the special case $[A, [A, B] = [B, [A, B] = 0$, the equation

$$e^{\lambda A} e^{\lambda B} = e^{\lambda(A+B) + \frac{1}{2}\lambda^2 [A,B]} \tag{5}$$

holds. *Hint:* Show that both sides fulfill the same boundary condition for $\lambda = 0$, and then take the derivative of both sides with respect to λ to prove that both sides fulfill the same linear differential equation. By the theorem of the uniqueness of solutions of linear differential equations the two must then coincide.

Specialize to $\lambda = 1$ at the end.

Home Exercise 6 Coherent states of the Harmonic Oscillator

We turn back to the harmonic oscillator with creation and annihilation operators, $\hat{a}^\dagger$ und $\hat{a}$, with the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Furthermore, be $\lambda, \lambda^* \in \mathbb{C}$ two complex numbers complex conjugated to each other.

(a) Verify that the operator $U(\lambda, \lambda^*) = e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}$ is unitary,

$$U(\lambda, \lambda^*)^\dagger U(\lambda, \lambda^*) = U(\lambda, \lambda^*) U(\lambda, \lambda^*)^\dagger = \mathbb{1}. \tag{6}$$

(b) Use the Baker-Campbell-Hausdorff formula from the previous exercise to show that

$$e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger}\, \hat{a}\, e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} = \hat{a} + \lambda \tag{7a}$$

$$e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger}\, \hat{a}^\dagger\, e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} = \hat{a}^\dagger + \lambda^*. \tag{7b}$$

How can you get the second equality from the first?

c) Just to make sure you remember it (we had it in sheet 1!), use BCH again to show:

$$e^{\lambda \hat{a}^\dagger \hat{a}}\, \hat{a}^\dagger\, e^{-\lambda \hat{a}^\dagger \hat{a}} = \hat{a}^\dagger e^{\lambda} \tag{8a}$$

$$e^{\lambda \hat{a}^\dagger \hat{a}}\, \hat{a}\, e^{-\lambda \hat{a}^\dagger \hat{a}} = \hat{a} e^{-\lambda} \tag{8b}$$

(d) Be $|0\rangle$ the normalized ground state of the harmonic oscillator, i.e. $\langle 0|0\rangle = 1$ and $\hat{a}\,|0\rangle = 0$. Define the coherent state now as $|\mu\rangle = e^{\mu \hat{a}^\dagger - \mu^* \hat{a}}\,|0\rangle$ für $\mu \in \mathbb{C}$.

Show that this is an eigenstate of the annihilation operator with complex eigenvalue μ:

$$\hat{a}\,|\mu\rangle = \mu\,|\mu\rangle \qquad . \tag{9}$$

Why are there eigenvectors with complex eigenvalues?

(e) Now show that

$$e^{\lambda a^\dagger - \lambda^* a}\,|\mu\rangle = |\lambda + \mu\rangle \qquad , \tag{10}$$

i.e. $|\lambda + \mu\rangle$ is eigenstate to $\hat{a}$ with eigenvalue $\lambda + \mu$.

Confirm that for arbitrary states,

$$\langle \lambda | v \rangle = \left\langle 0 \middle| e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger} \middle| v \right\rangle \tag{11}$$

holds, and for arbitrary operators $\hat{A}$

$$\left\langle \lambda \middle| \hat{A} \middle| \lambda \right\rangle = \left\langle 0 \middle| e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger}\, \hat{A}\, e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} \middle| 0 \right\rangle. \tag{12}$$

(f) Calculate the expectation values in the coherent state $|\lambda\rangle$:

$$\left\langle \hat{A} \right\rangle_{|\lambda\rangle} = \frac{\left\langle \lambda \middle| \hat{A} \middle| \lambda \right\rangle}{\langle \lambda | \lambda \rangle} \tag{13}$$

for the following operators

$$\langle \hat{a} \rangle_{|\lambda\rangle}\,,\ \left\langle \hat{a}^\dagger \right\rangle_{|\lambda\rangle}\,,\ \langle \hat{x} \rangle_{|\lambda\rangle} = \frac{1}{\sqrt{2}} \left\langle (\hat{a} + \hat{a}^\dagger) \right\rangle_{|\lambda\rangle}\,,\ \langle \hat{p} \rangle_{|\lambda\rangle} = \frac{1}{\sqrt{2}\,\mathrm{i}} \left\langle (\hat{a} - \hat{a}^\dagger) \right\rangle_{|\lambda\rangle}\,.$$

(You first need the normalization of the coherent states, obviously).

(g) Show that the for the Hamiltonian of the harmonic oscillator the states

$$|\lambda, t\rangle = \exp[-\mathrm{i}\omega t \hat{a}^\dagger \hat{a}]\, |\lambda\rangle \tag{14}$$

solve the Schrödinger equation. Why can you drop the contribution from the ground state energy?

Calculate the time-dependent expectation values

$$\left\langle \hat{A} \right\rangle_{|\lambda,t\rangle} = \frac{\left\langle \lambda, t \middle| \hat{A} \middle| \lambda, t \right\rangle}{\langle \lambda, t | \lambda, t \rangle} \tag{15}$$

for

$$\langle \hat{x} \rangle_{|\lambda,t\rangle} = \frac{1}{\sqrt{2}} \left\langle (\hat{a} + \hat{a}^\dagger) \right\rangle_{|\lambda,t\rangle} \qquad \text{and} \qquad \langle \hat{p} \rangle_{|\lambda,t\rangle} = \frac{1}{\sqrt{2}\,\mathrm{i}} \left\langle (\hat{a} - \hat{a}^\dagger) \right\rangle_{|\lambda,t\rangle} \tag{16}$$

(h) Determine the uncertainties of position and momentum in the time-dependent coherent states:

$$\left\langle (\Delta x)^2 \right\rangle_{|\lambda,t\rangle} = \left\langle \hat{x}^2 \right\rangle_{|\lambda,t\rangle} - \langle \hat{x} \rangle^2_{|\lambda,t\rangle} \quad \text{und} \quad \left\langle (\Delta p)^2 \right\rangle_{|\lambda,t\rangle} = \left\langle \hat{p}^2 \right\rangle_{|\lambda,t\rangle} - \langle \hat{p} \rangle^2_{|\lambda,t\rangle} \tag{17}$$

Remark: The proper uncertainties are the square roots of these expressions, the variances.

(i) Interpret the results and compare with the classical system.

(j) Show that the coherent states can be written as:

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} e^{\beta \hat{a}^\dagger} |0\rangle\,, \tag{18}$$

where $\beta \in \mathbb{C}$. *Hint:* Use Exercise 5 (c).

(k) Calculatue the coefficients c_n of the representation

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n\, |n\rangle \tag{19}$$

in terms of energy eigenstates of the oscillator and determine the distribution $|c_n|^2$ as a function of n. What is the mean value of this distribution?

Remark: When you know this distribution you can just read off the mean value.

(l) Show in the single-oscillator representation (19) for the coherent state $|\beta\rangle$ that it is an eigenstate of the annihilation operator a.