DESY Theory Group, Hamburg

Tutorials for the Lecture

Quantum Mechanics II

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Tutorial 1

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Home Exercise 1 (Anti-)Symmetrizer on N-Particle Hilbert space

In the lecture, it was shown that the following operator is a projector on the totally symmetric ($s = -1$) or totally antisymmetric ($s = +1$) sector of the N-particle Hilbert space:

$$\hat{S}_s = \sum_{\mathcal{P} \in S_N} \frac{(-s)^{\chi_{\mathcal{P}}}}{N!} \hat{U}_{\mathcal{P}} \qquad , \tag{1}$$

where $\mathcal{P}$ are permutations of N objects (the 1-particle Hilbert spaces here and the states therein), which are elements of the symmtric group S_N. $\chi_{\mathcal{P}}$ is the number of transpositions two mutual elements, out of which the permutation P consists, determining whether the permutation is even or odd. And $\hat{U}_{\mathcal{P}}$ is the unitary operator representing the permutation $\mathcal{P}$ on N-particle Hilbert space.

Show that this operator indeed has the properties of a projector, i.e. it is Hermitian, idempotent:

$$\hat{S}_s^\dagger = \hat{S}_s \qquad\qquad \hat{S}_s^2 = \hat{S}_s \qquad , \tag{2}$$

and fulfills

$$\hat{U}_{\mathcal{P}} \hat{S}_s = (-s)^{\chi_{\mathcal{P}}} \hat{S}_s \qquad . \tag{3}$$

Conclude that with a full sum over all permutations $\mathcal{P} \in S_N$, $\sum_{\mathcal{P} \in S_N}$ also $\sum_{\mathcal{P} \circ \mathcal{Q}}$ for $\mathcal{Q} \in S_N$ runs over the full group S_N. Why?

Home Exercise 2 Harmonic oscillators, creation/annihilaton operators

Hamiltonian for the harmonic oscillator:

$$\hat{\mathcal{H}} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \qquad [\hat{a}, \hat{a}^\dagger] = 1. \tag{4}$$

(a) Calculate $[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger]$ und $[\hat{a}^\dagger \hat{a}, \hat{a}]$. Assuming, $|n\rangle$ is eigenstate to $\hat{a}^\dagger \hat{a}$ withj eigenvalue n, show that $\hat{a}^\dagger |n\rangle$ eigenvector of $\hat{a}^\dagger a$ for the eigenvalue $(n+1)$ as well as $a|n\rangle$ (for $\hat{a}|n\rangle \neq 0$) eigenvector of $\hat{a}^\dagger \hat{a}$ for eigenvalue $(n-1)$.

(b) Determine $[\hat{a}^\dagger \hat{a}, (\hat{a}^\dagger)^n]$, $[\hat{a}^\dagger \hat{a}, \hat{a}^n]$ and the normalization of the eigenstates.

(c) Proof:

$$e^{\lambda \hat{a}^\dagger \hat{a}} \hat{a}^\dagger e^{-\lambda \hat{a}^\dagger \hat{a}} = \hat{a}^\dagger e^{\lambda}, \qquad e^{\lambda \hat{a}^\dagger \hat{a}} \hat{a} e^{-\lambda \hat{a}^\dagger \hat{a}} = \hat{a} e^{-\lambda}. \tag{5}$$

Hint: Do a Taylor expansion on both sides, and use part (a).

(d) Determine creation and annihilation operators in the Heisenberg picture, by explicit transformation as well by solving the Heisenberg equation of motion (for an arbitrary operator $\mathcal{O}$):

$$i\frac{\partial}{\partial t}\hat{\mathcal{O}} = [\hat{\mathcal{O}}, \hat{\mathcal{H}}]. \tag{6}$$

Home Exercise 3 3-dimensional harmonic oscillator

Be the Hamiltonian of the 3-dimensional harmonic oscillator given by

$$\hat{\mathcal{H}}_0 = \frac{1}{2m}\left(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2\right) + \frac{m}{2}\left(\omega_1^2\hat{x}_1^2 + \omega_2^2\hat{x}_2^2 + \omega_3^2\hat{x}_3^2\right) \tag{7}$$

(a) Express the position $\hat{x}_k$ and momentum operators $\hat{p}_k$, $k = 1, 2, 3$ with the help of the creation and annihilation operators

$$\begin{aligned} \hat{a}_k &= \frac{1}{\sqrt{2}}\left(\frac{1}{b_k}\hat{x}_k + i\frac{b_k}{\hbar}\hat{p}_k\right) \\ \hat{a}_k^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{1}{b_k}\hat{x}_k - i\frac{b_k}{\hbar}\hat{p}_k\right) \end{aligned} \tag{8}$$

where $b_k = \sqrt{\frac{\hbar}{m\omega_k}}$.

(b) Write the Hamiltonian $\hat{\mathcal{H}}_0$ as well as the orbital angular momentum operators $\hat{L}_i$, $i = 1, 2, 3$ with the help of the $\hat{a}_k$ and $\hat{a}_k^\dagger$.

(c) Determine for the special case $\omega_1 = \omega_2 = \omega_3 = \omega$ the eigenvectors $|n_1, n_2, n_3\rangle$ of $\hat{\mathcal{H}}_0$ for the energy eigenvalue $E = \frac{9}{2}\hbar\omega$.

(d) Now again for the general case of different frequencies, calculate the commutarors $\left[\hat{L}_i, \hat{\mathcal{H}}_0\right]$.