DESY Theory Group, Hamburg

Tutorials for the Lecture

# Standard Model of Particle Physics

WS 14/15

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**NOTE:** Exercise 1 and 2 are the derivations of the electroweak mass matrices and mixing given in the lectures. The important content of these exercises are 3 and 4, so you might first do them and exercises 1 and 2 afterwards. Also the latter need possibly less guidance.

**Exercise 1 The Electroweak Standard Model**

Consider a theory with the gauge group $SU(2)_L \times U(1)_Y$ and a complex scalar field $\phi$ that lies in the fundamental representation of $SU(2)$ and has charge $Y = 1/2$ under $U(1)$:

$$\phi(x) \to e^{i\alpha^a(x)T^a} e^{i\beta(x)Y} \phi(x), \tag{1}$$

where $T^a = \sigma^a/2$ with the usual Pauli matrices. If the field $\phi$ acquires a vacuum expectation value of the familiar form

$$\langle\phi\rangle \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2}$$

none of the four original generators annihilates the vacuum anymore.

a) Find the gauge transformation that leaves the vacuum invariant.

b) With the above definition of the transformation behavior of $\phi$ under the gauge group, its covariant derivative reads

$$D_\mu\phi \;=\; (\partial_\mu - igA^a_\mu T^a - ig' B_\mu Y)\phi. \tag{3}$$

The gauge boson mass terms arise from the square of (3) evaluated at the vacuum expectation value (2). The relevant terms are

$$\Delta\mathcal{L} = \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} \left( gA^a_\mu T^a + \frac{1}{2} g' B_\mu \right) \left( gA^{b\mu} T^b + \frac{1}{2} g' B^\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{4}$$

Evaluate the matrix product explicitly in order to obtain

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4} \left[ g^2 (A^1_\mu)^2 + g^2 (A^2_\mu)^2 + (-gA^3_\mu + g' B_\mu)^2 \right]. \tag{5}$$

There are now three massive vector bosons and one massless vector boson. In order to see this we identify

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left( A^1_\mu \mp A^2_\mu \right) \quad \text{with mass } \; m_W = g\frac{v}{2}. \tag{6}$$

Rewrite the square of the last term in (5) in terms of a scalar product

$$\frac{\sigma^b}{2} \begin{pmatrix} A^3_\mu & B_\mu \end{pmatrix} \mathcal{M} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}, \tag{7}$$

where $\mathcal{M}$ denotes the *mass matrix*. Find its eigenvalues and the corresponding eigenvectors. Define

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \tag{8}$$

and calculate $\cos\theta_w$ and $\sin\theta_w$ in terms of the coupling constants $g,\ g'$.

c) Rewrite the covariant derivative in terms of the new fields (6), (8), the $SU(2)$ coupling constant $g$ and the *weak mixing angle* $\theta_w$. Identify the coupling constant of the massless gauge boson $A_\mu$ with the electric charge $e$ and the generators it couples to as

$$Q = T^3 + Y. \tag{9}$$

You should obtain

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) - i\frac{g}{\cos\theta_w} Z_\mu \left(T^3 - \sin^2\theta_w\, Q\right) - ieA_\mu Q, \tag{10}$$

where $T^\pm = T^1 \pm iT^2$.

## Exercise 2 Electroweak Gauge Interactions with fermions

The covariant derivative (10) uniquely defines the coupling of the gauge bosons to fermions, once the quantum numbers are specified. To determine those quantum numbers, one has to take two experimentally well established facts into account:

a) the $W$ boson only couples to left-handed fermions

b) the electric charge of the fermions.

Implementing condition a) is straightforward since the kinetic term of a Dirac fermion splits into separate pieces for left- and right-handed fields:

$$\bar{\psi} i\partial\!\!\!/ \psi = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R. \tag{11}$$

When coupling $\psi$ to gauge fields, one can simply choose $\psi_L$ and $\psi_R$ do be in different representations of the gauge group. I.e. we choose the left-handed fields to transform as doublets, whereas the right-handed fields transform trivially under $SU(2)_L$. Once we have specified a the $T^3$ value for each fermion field, we have to choose its $U(1)_Y$ hypercharge such that its electric charge satisfies (9). For example, the charge assignments for one quark generation looks like

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}_{Y=1/6}, \quad (u_R)_{Y=2/3}, \quad (d_R)_{Y=-1/3}. \tag{12}$$

a) With this at hand, deduce the charge assignments for the $\nu, e$ lepton fields.

b) Convince youself that the above choices of the charges do not allow for mass terms:

$$\mathcal{L}_m = -m_e\left(\bar{e}_L e_R + \bar{e}_R e_L\right). \tag{13}$$

c) Show that the electroweak currents have the following structure for the SM fermions

$$\begin{aligned}\mathcal{L} &= \bar{E}_L(i\partial\!\!\!/)E_L + \bar{e}_R(i\partial\!\!\!/)e_R + \bar{Q}_L(i\partial\!\!\!/)Q_L + \bar{u}_R(i\partial\!\!\!/)u_R + \bar{d}_R(i\partial\!\!\!/)d_R \\ &\quad + g\left(W^+_\mu J^{\mu+}_W + W^-_\mu J^{\mu-}_W + Z_\mu J^\mu_Z\right) + eA_\mu J^\mu_{\text{em}},\end{aligned} \tag{14}$$

where

$$
\begin{aligned}
J_W^{\mu+} &= \left(\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L\right); \\
J_W^{\mu-} &= \left(\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L\right); \\
J_Z^\mu &= \frac{1}{\cos\theta_w}\left[\bar{\nu}_L \gamma^\mu \left(\frac{1}{2}\right)\nu_L + \bar{e}_L \gamma^\mu \left(-\frac{1}{2} + \sin^2\theta_w\right) e_L + \bar{e}_R \gamma^\mu \left(\sin^2\theta_w\right) e_R \right. \\
&\quad + \bar{u}_L \gamma^\mu \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_L + \bar{u}_R \gamma^\mu \left(-\frac{2}{3}\sin^2\theta_w\right) u_R \\
&\quad \left. + \bar{d}_L \gamma^\mu \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_w\right) d_L + \bar{d}_R \gamma^\mu \left(\frac{1}{3}\sin^2\theta_w\right) d_R \right]; \\
J_{\text{em}}^\mu &= \bar{e}\gamma^\mu(-1)e + \bar{u}\gamma^\mu(2/3)u + \bar{d}\gamma^\mu(-1/3)d.
\end{aligned}
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## Exercise 3 Polarization vectors of massive vector particles

From the three polarization vectors $\epsilon_\mu^{(\lambda)}$ of a massive vector boson, the polarization tensor is constructed according to

$$P_{\mu\nu} = \sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)*}.$$

Starting in the rest frame $k_\mu = (m, 0)$, where the polarization vectors are given as the Cartesian basis, show via a Lorenz transformation that

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}.$$

Hints:

(a) For simplification, perform a boost along the $z$-axis.

(b) Express the Lorenz transformations $\Lambda^\mu_\nu$ solely in terms of the energy $\omega$ and the modulus of the momentum $|\vec{k}|$.

## Exercise 4 Widths of the electroweak vector bosons

Calculate the widths of the $W$ and $Z$ boson. **hint:** Calculate the partial widths for the decays $Z \to f\bar{f}$ and $W \to f_i\bar{f}_j$ and sum over all possible final states. How do the color charges of the quarks contribute? Neglect the fermion masses and express the widths in terms of the Fermi constant $G_F/\sqrt{2} = g^2/(8M_W^2)$. What are the numerical results? You can cross-check your results at `http://pdg.lbl.gov/`.