DESY Theory Group, Hamburg

Tutorials for the Lecture

Standard Model of Particle Physics

WS 14/15

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Tutorial 3

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Exercise 1 D-dim Gamma matrices

In order to calculate fermion loops using dimensional regularization, one has to generalize the Dirac matrices to arbitrary space-time dimensions. Let $g^{\mu\nu}$ be $D-$dim, e.g. $\mu, \nu = 0, 1, \ldots, D-1$ with

$$\delta^\mu_\mu = g^\mu_\mu = D \quad \Rightarrow \quad \#(\gamma^\mu) = D.$$

The defining Dirac algebra remains unchanged: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$.

Show that

$$\gamma^\mu\gamma_\mu = D\mathbb{1}_4, \quad \gamma^\mu \not{a} \gamma_\mu = (2-D)\not{a} \tag{1}$$

Exercise 2 Feynman parameters

Show:

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \, \frac{1}{[ax + b(1-x)]^2} = \int_0^1 \mathrm{d}x \, \mathrm{d}y \; \delta(1-x-y) \frac{1}{[ax+by]^2} \tag{2}$$

$$\frac{1}{ab^n} = \int_0^1 \mathrm{d}x \, \mathrm{d}y \; \delta(1-x-y) \frac{ny^{n-1}}{[ax+by]^{n+1}} \tag{3}$$

$$\frac{1}{a_1 \ldots a_n} = \int_0^1 \left(\prod_{i=1}^n \mathrm{d}x_i \right) \; \delta(1 - \sum_{i=1}^n x_i) \frac{(n-1)!}{[a_1 x_1 + \ldots + a_n x_n]^n} \tag{4}$$

$$\frac{1}{a_1^{m_1} \ldots a_n^{m_n}} = \int_0^1 \left(\prod_{i=1}^n \mathrm{d}x_i \right) \; \delta(1 - \sum_{i=1}^n x_i) \frac{\prod_{i=1}^n x_i^{m_i - 1}}{[\sum_{i=1}^n x_i a_i]^{\Sigma m_i}} \frac{\Gamma(m_1 + \ldots + m_n)}{\Gamma(m_1) \ldots \Gamma(x_n)} \tag{5}$$

Hint: Proof (3) by induction.

Exercise 3 Beta function

Given the two integral representation of the Euler Beta and Gamma functions

$$\Gamma(x) = \int_0^\infty \mathrm{d}t \; t^{x-1} e^{-t}, \qquad B(x,y) = \int_0^1 \mathrm{d}t \; t^{x-1}(1-t)^{y-1}. \tag{6}$$

proof the connection between them:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{7}$$

Exercise 4 Scalar Master Integrals

With the expansion of the Γ-function around 0

$$\lim_{z\to 0}\Gamma(z) = \frac{1}{z} - \gamma_E + \mathcal{O}(z)$$

show that

a) the scalar 1-point function is given by (use D-dim. spherical coordinates for the momentum integration, show that the D-dim. solid angle is given by $2\pi^{D/2}/\Gamma(D/2)$ and use a proper substitution)

$$A_0(m) := \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\;(q^2-m^2+i\varepsilon)^{-1} = m^2\left[\Delta - \log\left(\frac{m^2}{\mu^2}\right)+1\right]+\mathcal{O}(D-4), \tag{8}$$

where $\Delta = \frac{2}{4-D} - \gamma_E + \log 4\pi$,

b) the scalar 2-point function is given by

$$B_0(p,m_0,m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\left\{\underbrace{(q^2-m_0^2+i\varepsilon)}_{=:a}\underbrace{[(q+p)^2-m_1^2+i\varepsilon]}_{=:b}\right\}^{-1} \tag{9}$$

$$= \Delta - \int_0^1 dx\;\log\left[\frac{x^2p^2 - x(p^2-m_1^2+m_0^2)+m_0^2-i\varepsilon}{\mu^2}\right] + \mathcal{O}(D-4) \tag{10}$$

c) the following symmetry exists: $B_0(p^2,m_0,m_1) = B_0(p^2,m_1,m_0)$

d) the special relation

$$B_0(p^2,0,m) = \Delta - \log\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2-p^2}{p^2}\log\left(\frac{m^2-p^2-i\varepsilon}{m^2}\right), \tag{11}$$

holds.

Exercise 5 Tensor Integrals

A general $N-$point tensor integral is given by

$$\begin{aligned}
&T^N_{\mu_1,\ldots,\mu_M}(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1})\\
&= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\;\frac{q_{\mu_1}\cdots q_{\mu_M}}{[q^2-m_0^2+i\epsilon][(q+p_1)^2-m_1^2+i\epsilon]\ldots[(q+p_{N-1})^2-m_{N-1}^2+i\epsilon]}\\
&\equiv \langle\frac{q_{\mu_1}\cdots q_{\mu_M}}{[q^2-m_0^2+i\epsilon][(q+p_1)^2-m_1^2+i\epsilon]\ldots[(q+p_{N-1})^2-m_{N-1}^2+i\epsilon]}\rangle_q
\end{aligned}$$

By convention, one denotes $T^1 = A,\ T^2 = B,\ T^3 = C,\ldots$. Looking at their Lorenz-structure one can find the following tensor reduction

$$B^\mu = p_1^\mu B_1, \tag{12}$$

$$B^{\mu\nu} = g^{\mu\nu}B_{00} + p_1^\mu p_1^\nu B_{11} \tag{13}$$

$$C^\mu = p_1^\mu C_1 + p_2^\mu C_2, \tag{14}$$

$$C^{\mu\nu} = \ldots. \tag{15}$$

a) Contract the relation from eq.(12) with p^μ to express B_μ as linear combination of the **scalar master integrals** A_0, B_0.

b) Similarly, show that

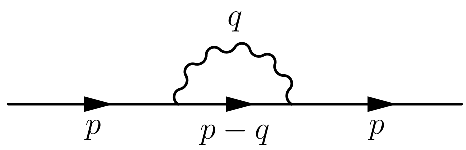
$$B_{00} = \frac{1}{2(D-1)} \left[A_0(m_1) + 2m_0^2 B_0 + (p^2 - m_1^2 + m_0^2) B_1\right], \quad (16)$$

$$B_{11} = \frac{1}{2(D-1)p^2} \left[(D-2)A_0(m_1) - 2m_0^2 B_0 - D(p^2 - m_1^2 + m_0^2) B_1\right]. \quad (17)$$

Homework Exercise 1 Electron Self-Energy

Calculate the electron self-energy at one loop

$\Sigma^{\bar{e}e}(p) =$



$$= ie^2 \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\gamma_\mu \left(\not{q} + m\right) \gamma^\mu}{[q^2 - \lambda^2] \left[(q-p)^2 - m^2\right]}. \quad (18)$$

λ denotes in this context a photon mass which is introduced in order to regularize the soft divergence of this diagram. It does not appear in physical observables. You should just express the result in terms of scalar master integrals.