DESY Theory Group, Hamburg

Tutorials for the Lecture

Standard Model of Particle Physics WS 14/15

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Tutorial 2
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Exercise 1 Non-Abelian Lie Algebras

Show that in general matrices fulfill:

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = 0$$

Use the standard definition of the structure constants in a Lie algebra $\left[T^a, T^b\right] = i f^{abc} T^c$ to show that they fulfill the Jacobi identity (repeated indices are again summed over):

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0.$$

Show that this implies that the structure constants themselves are a representation of the Lie algebra (the adjoint representation).

How does the quadratic Casimir operator of the adjoint representation look like.

For the color algebra, use:

$$\text{Tr}\left[T^a T^b\right] = T_F \delta^{ab},$$

with Dynkin index $T_F = 1/2$ for the fundamental representation.

Prove

$$T^b T^a T^b = (C_F - \frac{1}{2} C_A) T^a$$

with the quadratic Casimir operators $C_F = 4/3$, $C_A = 3$ for $SU(3)$.
What gives Tr $[T^a T^b T^a T^b]$ then?

Exercise 2 Resonance production of massive vector bosons and collinear photon radiation

Add to QED a massive photon field B_μ of mass M, which couples to electrons via

$$\Delta H = \int d^3x \left(g \bar{\psi} \gamma^\mu \psi B_\mu\right).$$

A massive photon in the initial or final state has three physical polarizations, corresponding to three space-like unit vectors in the boson's rest frame. These can be characterized invariantly, in terms of the boson's 4-momentum k_μ, as three vectors $\epsilon_\mu^{(i)}$ satisfying

$$\epsilon^{(i)} \epsilon^{(j)} = -\delta^{ij}, \quad k^\mu \epsilon_\mu^{(i)} = 0.$$

The four vectors $(k_\mu/M, \epsilon_\mu^{(i)})$ form a complete orthonormal basis. Since B_μ couples to the conserved current $\bar{\psi}\gamma^\mu\psi$, the Ward identity implies that k_μ dotted into the amplitude for B production yields zero; thus we can replace:

$$\sum_i \epsilon_\mu^{(i)} \epsilon_\nu^{(i)} \to -g_{\mu\nu}.$$

This gives a generalization to massive bosons of the Feynman trick for photon polarizations and simplifies the calculation of B production cross sections. (Warning: This trick does not work (so simply) for non-Abelian gauge fields.) In the following calculations a vanishing electron mass shall be assumed.

a) Compute the cross section for the process $e^+e^- \to B$. Compute the lifetime of the B, assuming that it decays only to electrons. Verify the relation

$$\sigma(e^+e^- \to B) = \frac{12\pi^2}{M}\Gamma(e^+e^- \to B)\delta(M^2 - s).$$

b) Compute the differential cross section, in the center-of-mass system, for the process $e^+e^- \to \gamma + B$. (This calculation goes over almost unchanged to the realistic processes $e^+e^- \to \gamma + \Upsilon$ or $e^+e^- \to \gamma + Z$; this allows one to measure the number of Z decays into unobserved final states, which in turn is proportional to the number of neutrino species.)

c) Notice that the cross section of part b) diverges as $\theta \to 0$ or π. Let us analyze the region near $\theta = 0$. In this region, the dominant contribution comes from the t-channel diagram and corresponds intuitively to the emission of a photon from the electron line before e^+e^- annihilation into a B. Let us rearrange the formula in such a way to support this interpretation. First, note that the divergence as $\theta \to 0$ is cut off by the electron mass: Let the electron momentum be $p^\mu = (E, 0, 0, k)$, with $k = \sqrt{E^2 - m^2}$, and let the photon momentum be $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$.

   Show that the denominator of the propagator never becomes smaller than $\mathcal{O}(m^2/s)$: Now integrate the cross section of part b) over forward angles, cutting off the θ integral at $\theta^2 \approx \mathcal{O}(m^2/s)$ and keeping only the leading logarithmic term, proportional to $\log(s/m^2)$. Show that, in this approximation, the cross section for forward photon emission can be written as

$$\sigma(e^+e^- \to \gamma + B) \approx \int dx\, f(x)\, \sigma(e^+e^- \to B \text{ at } E_{cm}^2 = (1-x)s),$$

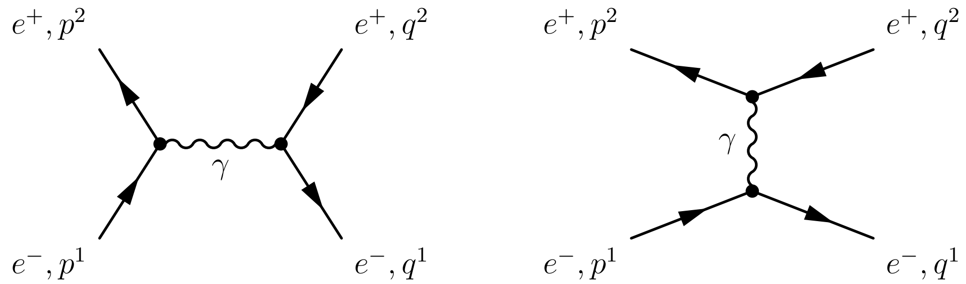
   where the annihilation cross section is evaluated for the collision of a positron of energy E with an electron of energy $(1-x)E$, and the function $f(x)$, the *Weizsäcker-Williams distribution function*, is given by

$$f(x) = \frac{\alpha}{2\pi}\frac{1 + (1-x)^2}{x}\log\left(\frac{s}{m^2}\right).$$

   This function arises universally in processes in which a photon is emitted collinearly from an electron line, independent of the subsequent dynamics.

Homework Exercise 1 Bhabha scattering

Consider Bhabha scattering, the scattering of an electron with a positron, $e^+e^- \to e^+e^-$. There are two Feynman diagrams:



(a) Determine the amplitude using the Feynman rules of QED. Note the relative sign between the two amplitudes! Where does it come from again? Using the approximation $\sqrt{s} \gg m_e$ to neglect the electron mass completely, calculate the squared spin-averaged matrix element.

[**8 points**]

(b) Insert the Mandelstam variables to show that the differential cross section can be written as

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] \tag{1}$$

Show that the squared amplitude is symmetric under exchange of $s \leftrightarrow t$. Why?

[**6 points**]

(c) Express the differential cross section by $\cos\theta$ aus and plot it as a function of $\cos\theta$. Which property of the Feynman diagrams causes the divergence of the cross section for $\cos\theta \to 0$?

[**4 points**]