DESY Theory Group, Hamburg

Tutorials for the Lecture

Standard Model of Particle Physics

WS 14/15

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Tutorial 1
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Home Exercise 1 $2 \to 2$ **processes, Mandelstam variables**

For scatterings of elementary particles with two particles in the initial and final state each ($2 \to 2$ processes) with

$$p_1 + p_2 \to q_1 + q_2, \quad p_1^2 = m_1^2,\ p_2^2 = m_2^2,\ q_1^2 = m_3^2,\ q_2^2 = m_4^2 \tag{1}$$

we introduce the relativistically invariant Mandelstam variables:

$$s \equiv (p_1 + p_2)^2 = (q_1 + q_2)^2 \tag{2a}$$
$$t \equiv (q_1 - p_1)^2 = (p_2 - q_2)^2 \tag{2b}$$
$$u \equiv (q_1 - p_2)^2 = (p_1 - q_2)^2 \tag{2c}$$

a) Show:

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{3}$$

[**2 points**]

b) Be $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$. Show that the momentum in the center-of-mass system can be written as

$$\vec{p}^{\,2}_{\text{CMS}} = \frac{1}{4s}\lambda(s, m_1^2, m_2^2). \tag{4}$$

[**3 points**]

Home Exercise 2 Two-particle phase space

Consider an arbitary $2 \to 2$ process with the usual kinematics: $p_1 + p_2 \to q_1 + q_2$, the differential cross section

$$d\sigma = \frac{1}{4\sqrt{(p_1 p_2)^2 - p_1^2 p_2^2}}(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)\widetilde{dq}_1 \widetilde{dq}_2 \left|\mathcal{M}\right|^2 \tag{5}$$

and the invariant integration measure:

$$\widetilde{dk} \equiv \frac{d^4 k}{(2\pi)^4}\Theta(k_0)(2\pi)\delta(k^2 - m^2) = \left.\frac{d^3\vec{k}}{(2\pi)^3 2k_0}\right|_{k_0 = \sqrt{\vec{k}^2 + m^2}}. \tag{6}$$

a) Show that in the center-of-mass system Eq.(5) yields

$$\frac{d\sigma}{d\Omega_{\text{CMS}}} = \frac{|\vec{q}_{\text{CMS}}|}{|\vec{p}_{\text{CMS}}|}\frac{1}{64\pi^2 s}\left|\mathcal{M}\right|^2 \tag{7a}$$
$$\frac{d\sigma}{dt} = \frac{1}{64\pi s \vec{p}^{\,2}_{\text{CMS}}}\left|\mathcal{M}\right|^2 \tag{7b}$$

Here, $t \equiv (p_1 - q_1)^2 = (p_2 - q_2)^2$ is one of the Mandelstam variables defined above.

[**6 points**]

b) How do the formulae simplify in the case of four massless scattering particles?

[**2 points**]

Home Exercise 3 Properties of Gamma matrices

(a) Use the so called chiral representation of the Gamma matrices:

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{8}$$

Show that they indeed obey the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \times \mathbb{1}$.

[**3 points**]

(b) Show: γ^0 is Hermitian, γ^i anti-Hermitian.

[**2 points**]

(c) Proof

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \qquad \gamma^0(S^{\mu\nu})^\dagger\gamma^0 = S^{\mu\nu} \qquad , \tag{9}$$

where $S = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is the generator of Lorentz boosts and rotations.

[**4 points**]

(d) The matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is important for polarization and in the electroweak theory. Plug in one index combination and use symmetry properties to argue that

$$\gamma^5 = \frac{-i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma, \qquad \text{mit} \quad \epsilon^{0123} = +1 \tag{10}$$

[**3 points**]

(e) Proof:

$$\{\gamma^\mu, \gamma^5\} = 0, \qquad [\gamma^5, S^{\mu\nu}] = 0 \tag{11}$$

[**3 points**]

(f) Determine γ^5 in the chiral representation. Which meaning do the matrices $\frac{1}{2}(1 \pm \gamma^5)$ have?

[**3 points**]

Home Exercise 4 More properties of Gamma matrices

(a) Never use a definite representation of the gamma-matrices in this exercise. The following identities are important for the calculation of processes in the Standard Model.

Show, that the trace of an odd number of gamma matrices vanishes. (For this insert unity in the form $\gamma^5\gamma^5$. (Show that $(\gamma^5)^2 = 1$. What does this mean for Tr γ^μ?

[**3 points**]

(b) Show Tr $\gamma^5 = 0$.

[**3 points**]

(c) Use the Dirac algebra to show that

$$\text{Tr}\ \gamma^\mu\gamma^\nu = 4g^{\mu\nu} \qquad \text{Tr}\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}). \tag{12}$$

[**5 points**]

(d) Proof Tr $\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_n} = 0$ for odd n.

[**2 points**]

(e) Proof: Tr $\gamma^\mu\gamma^\nu\gamma^5 = 0$. (The square of which gamma matrix you need to insert here?)

[**3 points**]

(f) Calculate Tr $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5$ by using the symmetry in the four indices and then a fixed combination.

[**3 points**]

(g) Proof the contraction identitites for gamma matrices:

$$\gamma^\mu\gamma_\mu = 4 \quad \gamma^\mu\gamma^\rho\gamma_\mu = -2\gamma^\rho \quad \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4g^{\rho\sigma} \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{13}$$

[**4 points**]

Home Exercise 5 Trace with inverted order

Again the chiral representation: Show that the matrix $\mathcal{C} = i\gamma^2\gamma^0$ has the following properties:

$$\mathcal{C}^{-1} = \mathcal{C}^T = -\mathcal{C} \qquad \mathcal{C}\,(\gamma^\mu)^T\,\mathcal{C}^{-1} = -\gamma^\mu. \tag{14}$$

Now use this last property and the results from exercise 4 a) as well as Tr M = Tr M^T to show:

$$\text{Tr}\ \gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n} = \text{Tr}\ \gamma^{\mu_n}\ldots\gamma^{\mu_2}\gamma^{\mu_1}. \tag{15}$$

[**4 points**]