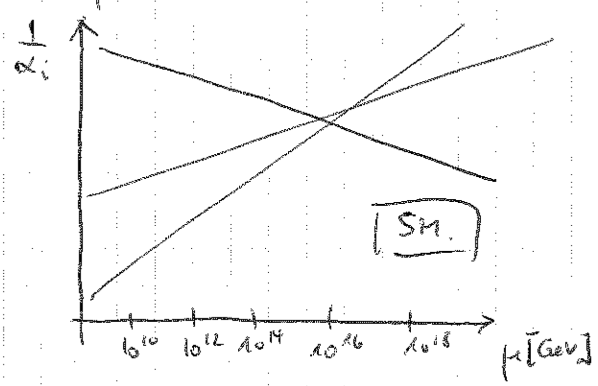
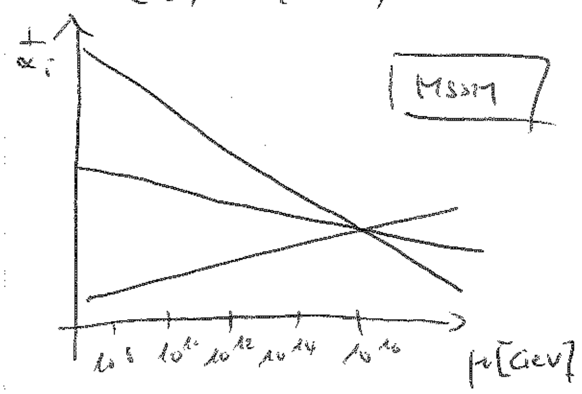
$\boxed{\text{GUTs}}$

SUSY $\longrightarrow$ cf. SUSY literature (Drees et al., SUSY primer)
H. Georgi: Lie algebras in particle physics, Harvard Univ. Press, 1992
R. Slansky: Group Theory for Unified Model Building, Phys. Rep. 79 (1981), 1
R. Mohapatra: Unification and SUSY, Springer, 1986
P. Langacker: GUTs, Phys. Rep. 72 (1981), 185
P. Nath / P. Fileviez Perez, Proton Stability, hep-ph / 0601023

Running of gauge couplings

$$\frac{dg_a}{d\ln\mu} = \frac{g_a^3}{16\pi^2}\, b_a \qquad \begin{array}{ll} \text{SM} & b_a = \left(\frac{41}{10}, -\frac{19}{6}, -7\right) \\ \text{MSSM} & b_a = \left(\frac{33}{5}, 1, -3\right) \end{array}$$





Prime example: $SU(5)$ Georgi / Glashow, 1973.

$$SU(5) \xrightarrow[M_X]{} SU(3)_c \times SU(2)_L \times U(1)_Y \xrightarrow[M_Z]{} SU(3)_c \times U(1)_{em.}$$

$SU(5)$ has $5^2-1 = 24$ generators:

$$24 \longrightarrow \underbrace{(8,1)_0}_{G_\mu^a} \oplus \underbrace{(1,3)_0}_{W_\mu^{\pm,0}} \oplus \underbrace{(1,1)_0}_{B_\mu} \oplus \underbrace{(3,2)_{\frac{5}{3}}}_{X_\mu, Y_\mu} \oplus \underbrace{(\bar{3},2)_{-\frac{5}{3}}}_{X_\mu^c, Y_\mu^c}$$

whole $SU(5)$ gauge field:

$$A_\mu = \sum_{a=1}^{24} A_\mu^a \frac{\lambda^a}{2} = \frac{1}{\sqrt{2}} \begin{array}{c} \left(\begin{array}{c|cc} \sqrt{2}\, G_\mu^a \frac{\lambda^a_{\text{Gell-Mann}}}{2} & \bar{X}_\mu^r & \bar{Y}_\mu^r \\ & \bar{X}_\mu^g & \bar{Y}_\mu^g \\ & \bar{X}_\mu^b & \bar{Y}_\mu^b \\ \hline X_\mu^r \;\; X_\mu^g \;\; X_\mu^b & & \\ Y_\mu^r \;\; Y_\mu^g \;\; Y_\mu^b & \multicolumn{2}{c}{\sqrt{2}\, W_\mu^a \frac{\sigma^a}{2}} \end{array}\right) \\ \text{red} \;\; \text{green} \;\; \text{blue} \end{array} \begin{array}{l} \text{red} \\ \text{green} \\ \text{blue} \end{array}$$

$$- \frac{1}{\sqrt{2}\cdot\sqrt{15}}\, B_\mu \left(\begin{array}{ccc|cc} -2 & & 0 & & \\ & -2 & & & 0 \\ 0 & & -2 & & \\ \hline & 0 & & +3 & 0 \\ & & & 0 & +3 \end{array}\right)$$

Predictions on quantum numbers:

* Hyper-charge: $\frac{\lambda_{12}}{2} = \sqrt{\frac{3}{5}}\, \frac{Y}{2}$, $Y = \text{diag}(-2, -2, -2, 3, 3)$

Quantized hyper-charges are fixed by non-Abelian generator

Factor of $\sqrt{\frac{3}{5}}$ comes from normalization of operator, usually $Tr[T_R^a T_R^b] = d_R \cdot \delta^{ab}$, $d_R \equiv \frac{1}{2}$ Dynkin index of representation R. (usually 1/2). Summing up hyperchargs leads to the conversion factor of 3/5 between $T_{12} T_{12}$ and $Y \cdot Y$.

* Weak Isospin: $T_{1/2/3}/2 = \frac{\lambda_{9,10,11}}{2}$.
* Strong interactions: $\lambda^{\text{Gell-Mann}}_{1,\dots,8}/2 = \frac{\lambda_{1,\dots,8}}{2}$
* Electric charge: $Q = \frac{T^3}{2} + \frac{Y}{2} = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0\right)$

(for the fundamental representation to follow).

* Weak mixing angle: $\frac{d\alpha_i}{d\ln\mu} = b_i \frac{\alpha_i^2}{2\pi}$

SM, coefficients:

$$b_3 = -11 + \frac{4}{3} N_{gen.}$$
$$b_2 = -\frac{22}{3} + \frac{4}{3} N_{gen} + \frac{1}{6} N_{Higgs}$$
$$b_1 = \frac{20}{9} N_{gen} + \frac{1}{6} N_{Higgs}$$

$$\Longrightarrow \quad \frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(\mu_0)} - \frac{b_i}{4\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

$$\alpha_1(\mu) = \frac{\alpha(\mu)}{\cos^2\theta_W(\mu)} \equiv \frac{\alpha(\mu)}{c_W^2(\mu)}, \quad \alpha_2(\mu) = \frac{\alpha(\mu)}{\sin^2\theta_W(\mu)} = \frac{\alpha^2(\mu)}{s_W^2(\mu)}$$

is also written as

$\frac{4}{3} = \frac{20}{9} \cdot \frac{3}{5}$ $\qquad$ $\frac{1}{10} = \frac{3}{5} \cdot \frac{1}{6}$

$\boxed{\text{SM}}$
$$b_i = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix} + N_{Gen.} \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + N_{Higgs} \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix}$$

$\boxed{\text{MSSM}}$
$$b_i = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + N_{Gen.} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + N_{Higgs} \begin{pmatrix} 3/10 \\ 1/2 \\ 0 \end{pmatrix}$$

Wino / Gluino contribution

general formula:
$$\boxed{b_i = -\frac{11}{3} C_A + \frac{2}{3} N^{\text{chiral}}_{\text{ferm.}} T(R) + \frac{1}{3} N_{\text{sc.}} T(R)}$$

quadratic Casimir: N for SU(N) $\qquad$ $\frac{1}{2}$ for SU(N)

SU(2): $\quad Q, L \Rightarrow \frac{1}{2}\left(\frac{2}{3} \cdot \underbrace{3}_{\text{color}} + \frac{2}{3} \cdot 1\right) = \frac{4}{3}$, $\quad$ SU(3) $= \frac{1}{2} \cdot \frac{2}{3}(2 + 1 + 1) = \frac{4}{3}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $Q \;\; u^c \;\; d^c$

$$U(1)_Y = \frac{1}{4}\Big(\underbrace{3 \cdot 2 \left(\tfrac{1}{3}\right)^2}_{Q} + \underbrace{2 \cdot (-1)^2}_{L} + \underbrace{3 \cdot \left(\tfrac{4}{3}\right)^2}_{u^c} + \underbrace{3\left(\tfrac{2}{3}\right)^2}_{d^c} + \underbrace{(+2)^2}_{e^c}\Big) \cdot \frac{2}{3}$$

$$= \frac{1}{4} \cdot \frac{2}{3}\left(\frac{2}{3} + 2 + \frac{16}{3} + \frac{4}{3} + 4\right) = \frac{1}{9}(1 + 3 + 8 + 2 + 6) = \frac{20}{9}.$$

(Gauge) Unification and the running of couplings

VOLUME 32, NUMBER 8 PHYSICAL REVIEW LETTERS 25 FEBRUARY 1974

Unity of All Elementary-Particle Forces

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(Received 10 January 1974)

Strong, electromagnetic, and weak forces are conjectured to arise from a single fundamental interaction based on the gauge group SU(5).

We present a series of hypotheses and speculations leading inescapably to the conclusion that SU(5) is the gauge group of the world—that all elementary particle forces (strong, weak, and electromagnetic) are different manifestations of the same fundamental interaction involving a single coupling strength, the fine-structure constant. Our hypotheses may be wrong and our speculations idle, but the uniqueness and simplicity of our scheme are reasons enough that it be taken seriously.

of the GIM mechanism with the notion of colored quarks[4] keeps the successes of the quark model and gives an important bonus: Lepton and hadron anomalies cancel so that the theory of weak and electromagnetic interactions is renormalizable.[5]

The next step is to include strong interactions. We assume that *strong interactions are mediated by an octet of neutral vector gauge gluons* associated with local color SU(3) symmetry, and that there are no fundamental strongly interacting scalar-meson fields.[6] This insures that

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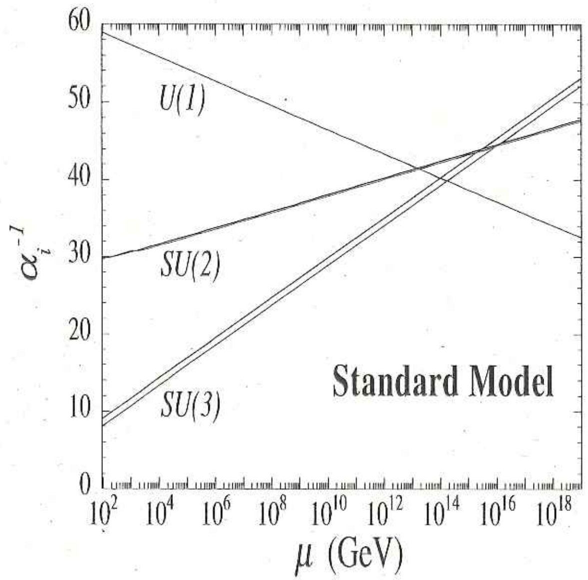
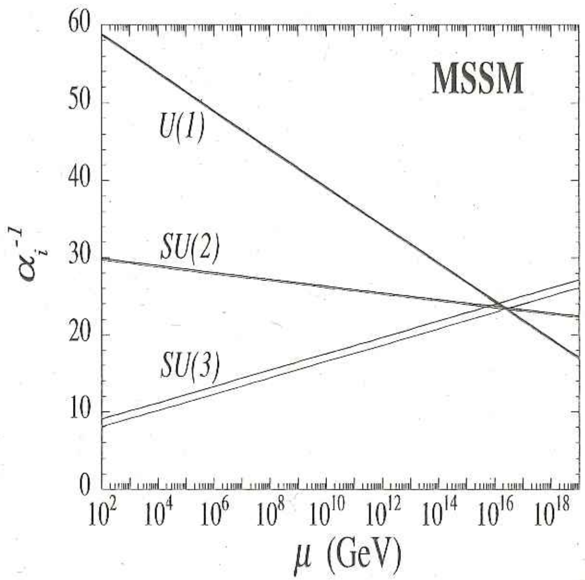
(Gauge) Unification and the running of couplings

Renormalization group (RG) running of gauge couplings:

$$\frac{dg_a}{d\log\mu} = \frac{g_a^3}{16\pi^2} B_a \qquad \begin{array}{ll} \text{SM} & B_a = \left(\frac{41}{10}, -\frac{19}{6}, -7\right) \\ \text{MSSM} & B_a = \left(\frac{33}{5}, 1, -3\right) \end{array}$$





VOLUME 32, NUMBER 8 PHYSICAL REVIEW LETTERS 25 FEBRUARY 1974

H. Georgi and H. D. Politzer, Phys. Rev. D (to be published); D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973), and Phys. Rev. D (to be published).

[16]A naive calculation indicates that the vector boson mass must be greater than 10^{15} GeV $\approx 10^{-9}$ g! Let the reader who finds this hard to swallow double the num-

31b

By definition: $\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT})$

$$\Rightarrow \frac{5}{3}\frac{\cancel{\alpha(M_{GUT})}}{c_W^2(M_{GUT})} = \frac{\cancel{\alpha(M_{GUT})}}{s_W^2(M_{GUT})} \Rightarrow \frac{s_W^2}{c_W^2} = \frac{3}{5}.$$

$$\Rightarrow s_W^2 = \frac{3}{5}(1-s_W^2) \Rightarrow \frac{8}{5}s_W^2 = \frac{3}{5} \Rightarrow \boxed{\sin^2\theta_W(M_{GUT}) = \frac{3}{8}}$$

$$\begin{aligned}
&(1) \quad \alpha_{GUT}^{-1} - \alpha_1^{-1}(\mu) = \frac{b_1}{2\pi}\ln(M_{GUT}/\mu)\\
&(2) \quad \alpha_{GUT}^{-1} - \alpha_2^{-1}(\mu) = \frac{b_2}{2\pi}\ln(M_{GUT}/\mu)\\
&(3) \quad \alpha_{GUT}^{-1} - \alpha_3^{-1}(\mu) = \frac{b_3}{2\pi}\ln(M_{GUT}/\mu)
\end{aligned}$$

Double ratio:

$$\frac{(2)-(3)}{(1)-(2)} = \xi \equiv \frac{b_2-b_3}{b_1-b_2} = \underbrace{\frac{\frac{\alpha(\mu)}{\alpha_s(\mu)} - s_W^2(\mu)}{s_W^2(\mu) - \frac{3}{5}c_W^2(\mu)}}_{(RHS)}$$

$$\Rightarrow \boxed{s_W^2(\mu) = \frac{3/5\,\xi}{1+8/5\,\xi} + \frac{1}{1+8/5\,\xi}\frac{\alpha(\mu)}{\alpha_s(\mu)}}$$

SM: $b_1 = 41/10$, $b_2 = -19/6$, $b_3 = -7$

MSSM: $b_1 = 33/5$, $b_2 = 1$, $b_3 = -3$.

$$\Rightarrow \xi_{SM} = \frac{115}{218}, \quad \xi_{MSSM} = \frac{5}{7}.$$

$$\Rightarrow \text{SM}: \quad \sin^2\theta_W(M_Z) = \frac{23}{134} + \frac{109}{201}\frac{\alpha(M_Z)}{\alpha_s(M_Z)} \simeq 0.207$$

$$\text{MSSM}: \quad \sin^2\theta(M_Z) = \frac{1}{5} + \frac{7}{15}\frac{\alpha(M_Z)}{\alpha_s(M_Z)} \simeq 0.231$$

Experimental value: $s_W^2(M_Z) \simeq 0.2312(3)$. ∇

Fermions (Matter [super] fields)

Only possible way to group them together:

$$\bar{5} = \begin{array}{|c|}\hline \\\hline \\\hline \\\hline \\\hline\end{array} = \begin{pmatrix} d_R^c \\ d_R^c \\ d_R^c \\ l_L \\ -\nu_L \end{pmatrix}\begin{array}{l} r\\ g\\ b\\ \\ \\ \end{array}$$

Remarks: $2 = \square = \bar{2}$, $\quad \bar{2} = \varepsilon\cdot 2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\nu_L\\ l_L\end{pmatrix} = \begin{pmatrix}l_L\\ -\nu_L\end{pmatrix}$

$$= i\sigma^2\cdot 2 =$$

$$5\otimes 5 = \square\otimes\square = \begin{array}{|c|}\hline \\\hline \\\hline\end{array} \oplus \begin{array}{|c|c|}\hline & \\\hline\end{array} = \underset{(\square\otimes\square)_a}{10} \oplus \underset{(\square\otimes\square)_s}{15}$$

~~Rk~~ $(3\otimes 3)_a = \bar{3}$ as in $(\square\otimes\square)_a = \begin{array}{|c|}\hline \\\hline \\\hline\end{array}$, $(\square\otimes\square)_s = \begin{array}{|c|c|}\hline & \\\hline\end{array}$

$$3\otimes 3 = \square\otimes\square = \begin{array}{|c|}\hline \\\hline \\\hline\end{array} \oplus \begin{array}{|c|c|}\hline & \\\hline\end{array} = \bar{3}\oplus 6$$

$$\overline{5} = (\overline{3},1)_{\frac{2}{3}} \oplus (1,2)_{-1}$$

$$10 = \boxminus = \frac{1}{\sqrt{2}} \begin{array}{cc} \left(\begin{array}{ccc|cc} 0 & u^c_b & u^c_g & -u_L & -d_L \\ -u^c_b & 0 & u^c_r & -u_L & -d_L \\ u^c_g & -u^c_r & 0 & -u_L & -d_L \\ \hline u_L & u_L & u_L & 0 & -e^c \\ d_L & d_L & d_L & e^c & 0 \end{array}\right) & \begin{array}{c} r \\ g \\ b \\ \\ \\ \end{array} \\ \begin{array}{ccccc} r & g & b & & \end{array} & \end{array}$$

$$10 = (3,2)_{\frac{1}{3}} \oplus (\overline{3},1)_{-\frac{4}{3}} \oplus (1,1)_2 \,.$$

Observations:

- Quarks and leptons in the same multiplet
- Fractional charges from tracelessness condition of GUT generators.
- $\overline{5}$ and 10 have equal and opposite anomalies.
- ν_R^c must be $SU(5)$ singlet.

Detour to other GUTS

SO(10) $SO(2N)$: fundamental (spinor) representation has dimension dim. $X = 2^{\frac{[N]}{2}}$ for N even

$SO(10)$: $2^4 = 16$ dim.

$Q_L, L_L, e^c_L, u^c, d^c, \nu^c$
$6 + 2 + 1 + 3 + 3 = 15 + 1 = 16$. now also contains right handed neutrinos.

maximal subgroups

$$SO(10) \to \begin{cases} SU(5) \times U(1) & SU(5) \\ SU(4) \times SU(2)_L \times SU(2)_R \ast & \text{Pati-Salam} \\ SU(5)' \times U(1) & \text{flipped } SU(5) \end{cases}$$

Classification of Lie algebras:

4 infinite families.
$SU(N)$ unitary algebras
$SO(2N+1)$ odd orthogonal algebras
$SO(2N)$ even orthogonal algebras
$Sp(2N)$ symplectic algebras.

and 5 special ones: G_2, F_4, E_6, E_7, E_8.

$SO(2N+1), SO(4N), Sp(2N), G_2, F_4, E_7, E_8$ have only real reps. (or pseudoreal reps.)

$SO(4N+2)$ (except $SO(2), SO(6)$), E_6 have complex reps., but they are all anomaly-free

$SU(N)$ not anomaly-free.

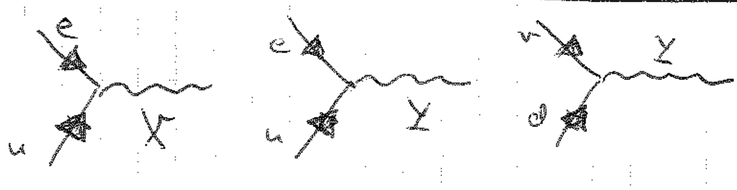
$\boxed{E_6}$: fundamental rep.: 27, adjoint reps. 78.

$E_6 \to SO(10) \times U(1)$ (one max. subgroup).

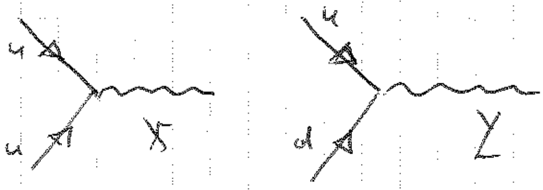
$\boxed{E_8} \longrightarrow E_6 \times SU(3)$ $\quad 248$ fundamental + adjoint rep.

$248 \longrightarrow 27_3 \oplus \overline{27}_{\bar{3}} \oplus 78_1 \oplus 1_8$.

$\boxed{\text{New interactions in GUTs.}}$

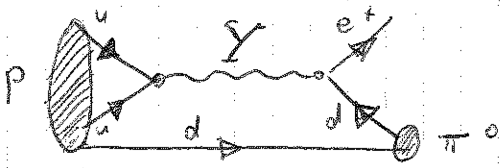


Leptoquark couplings.



Diquark couplings.

leads to:



$p \to e^+ \pi^0$

in(tro)duces proton decay.

$\boxed{\text{GUT symmetry breaking}}$

One can use a Higgs field in the adjoint represent.

$\Sigma \in 24$; $\quad \langle 0|\Sigma|0\rangle = w \cdot \mathrm{diag}\left(1,1,1,-\frac{3}{2},-\frac{3}{2}\right)$.

$M_X = M_Y = \frac{5}{2\sqrt{2}} g w$; $\quad SU(5) \xrightarrow{\langle 0|\Sigma|0\rangle} SU(3)_c \times SU(2)_L \times U(1)_Y$

There are other breaking mechanism, e.g. orbifold breaking, i.e. by geometric means / boundary conditions.

SM/MSSM Higgses must be embedded in irreps. (irreducible representations) of the GUT group (SU(5))

$$\mathbf{5} = \square = \begin{pmatrix} D_r \\ D_g \\ D_b \\ h^+ \\ h^0 \end{pmatrix}, \quad \bar{\mathbf{5}} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} = \begin{pmatrix} D^c_r \\ D^c_g \\ D^c_b \\ h^- \\ -h^0 \end{pmatrix}$$

$$\mathbf{5} = (3,1)_{-\frac{2}{3}} \oplus (1,2)_1 \qquad \bar{\mathbf{5}} = (\bar{3},1)_{\frac{2}{3}} \oplus \underset{\uparrow \text{SM Higgs}}{(1,2)_{-1}}$$

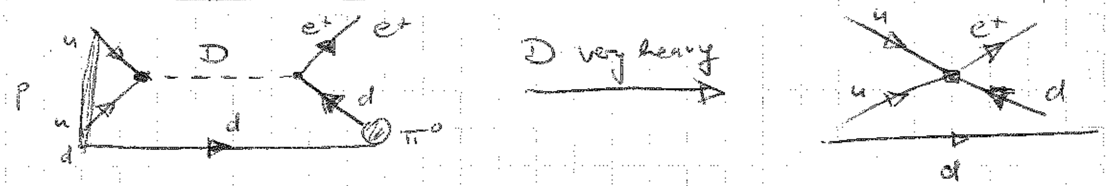
D, D^c: coloured triplet Higgses with charges $\pm\frac{1}{3}$.

D, D^c could also have leptoquark and diquark couplings, so could also induce proton decay, hence either the couplings have to be suppressed by orders of magnitude ($\cancel{\text{}}$ not possible because of Higgs couplings) or they have to be very heavy $\Longrightarrow$

$$m_H \sim 125 \text{ GeV}, \quad m_D \sim 10^{16} \text{ GeV} \Longrightarrow$$

known as $\boxed{\text{Doublet – Triplet Splitting Problem.}}$

(Naive) Estimate of Proton Lifetime



Effective 4-fermion operator (analog to muon decay)

$$\mathcal{L}_F = \frac{4G_F}{\sqrt{2}} (\bar{\mu} \gamma_\kappa \nu_\mu)(\bar{\nu}_e \gamma^\kappa e) \qquad \mathcal{L}_{GUT} = \frac{4 G_{GUT}}{\sqrt{2}} (\bar{u} \Gamma u)(\bar{e} \Gamma d)$$

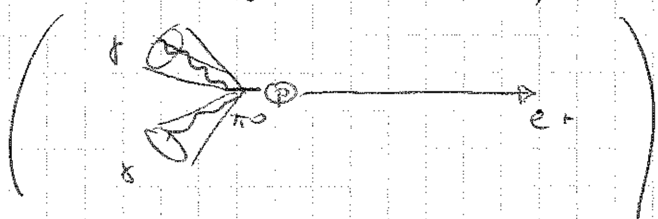
$$G_F / \sqrt{2} = g_2^2 / 8 M_W^2 \qquad \frac{G_{GUT}}{\sqrt{2}} = g^2 / 8 M_{GUT}^2$$

$$\tau(\mu \to e \nu_\mu \bar{\nu}_e) \sim \frac{192 \pi^3}{G_F^2 m_\mu^5} \qquad \tau(p \to e^+ \pi^0) \sim \frac{192 \pi^3}{G_{GUT}^2 m_p^5}$$

Proton lifetime for SUSY SU(5), i.e. $\alpha(M_{GUT}) \sim \frac{1}{24}$,

$$M_{GUT} \sim 2 \cdot 10^{16} \text{ GeV}; \quad \tau(p \to e^+ \pi^0) \sim \frac{M_{GUT}^4}{[\alpha(M_{GUT})]^2 m_p^5} \sim 10^{31 \pm 1} \text{ y}$$

Search for proton decays in Čerenkov detectors like Super-KamiokaNDE (50 kt RICH) mainly for



Photomultipliers.

HyperK (1 Mt), UNO (650 kt), European project Fréjus (1 Mt)

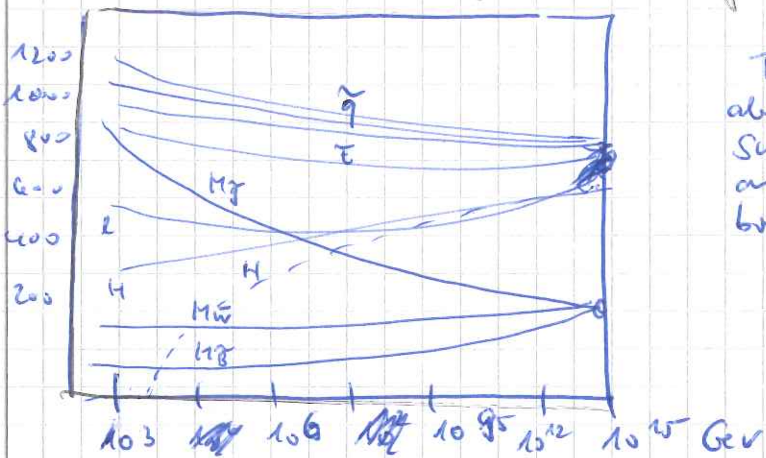
Precision of 10 yrs running $\Longrightarrow$ 10^{34}–10^{35} yrs sensitivity

Most sensitive channels:

$p \to e^+ \pi^0$ ca. ~~10~~ $1.6 \cdot 10^{33}$ yrs
$p \to e^+ \omega^0$ ca. $1 \cdot 10^{33}$ yrs
$p \to \mu^+ K^0$ ca. $1.3 \cdot 10^{33}$ yrs
$p \to \nu K^+$ ca. $2.3 \cdot 10^{33}$ yrs.

More issues / problems: Yukawa couplings have to obey also GUT unification conditions, which poses problems as top and tau are in the same multiplet.

In SUSY GUTs (stability of scales guaranteed by SUSY!)



This makes assumptions about the connection of SUSY breaking mechanism and the GUT symmetry breaking.

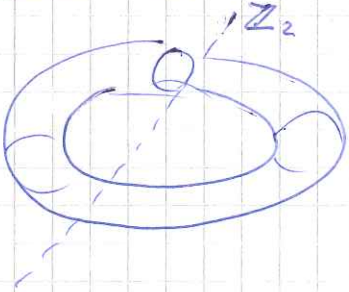
In general, there is also a relation for the gaugino masses, coupled / related to the corresponding running of the gauge couplings (which is true up to tiny 2-loop corrections).

$$\boxed{M_3 = \frac{\alpha_s}{\alpha} \sin^2\theta_W \cdot M_2 = \frac{3}{5}\frac{\alpha_s}{\alpha} \cos^2\theta_W \cdot M_1}$$

translating this to the TeV scale leads to $\boxed{M_3 : M_2 : M_1 \simeq 6 : 2 : 1}$

Note: in general, there can be large corrections from the GUT breaking, i.e. from integrating out heavy fields: threshold corrections.

Alternatives: Orbifold compactifications.



Torus: ~~[illegible]~~

$\mathbb{Z}_2$ parity identification leads to singular points, where there is a reduced gauge symmetry group.