Physikalisches Institut, Albert-Ludwigs-Universität Freiburg

Übungen zur Vorlesung

Quantenfeldtheorie

SS 09

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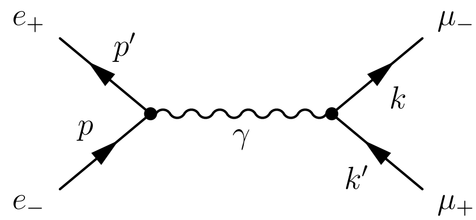
Blatt 11

Präsenz-Aufgabe 10 QED: $e^+ \, e^- \; \rightarrow \; \mu^+ \, \mu^-$



Calculate the cross section for the above process at tree-level using the formula

$$\frac{d\sigma}{d\Omega_{\text{CMS}}} = \frac{|\vec{k}_{\text{CMS}}|}{|\vec{p}_{\text{CMS}}|} \frac{1}{64\pi^2 s} |\mathcal{M}|^2 \tag{1}$$

and following the steps outlined below:

a) Derive the matrix element $\mathcal{M}$ using the Feynman rules given in the lecture.

b) The cross section is proportional to the absolut square of $\mathcal{M}$. Show that

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \left(\bar{v}(p')\gamma^\mu u(p)\, \bar{u}(p)\gamma^\nu v(p')\right) \left(\bar{u}(k)\gamma_\mu v(k')\, \bar{v}(k')\gamma_\nu u(k)\right). \tag{2}$$

c) Average $|\mathcal{M}|^2$ over spins in the initial state and sum over spins in the final state using the following completeness relations:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m. \tag{3}$$

What experimental setup does this correspond to?
The expression from eq.(2) should simplify to traces of products of γ matrices:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \text{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] \text{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]. \tag{4}$$

d) Evaluate the electron and muon trace from eq.(4). Now, neglect the electron mass and multiply the electron- with the muon- trace to obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4} \left[(pk)(p'k') + (pk')(p'k) + m_\mu^2 (pp')\right]. \tag{5}$$

e) To obtain a more explicit formula one has to choose a particular frame of reference - calculate $\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$ in the CMS with $\vec{p}, \vec{p}'$ along the z-axis and plug it into the formula for the differential cross section (eq.(1)):

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[\left(1+\frac{4m_\mu^2}{s}\right)+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right]. \tag{6}$$

f) Integrate over the solid angle to obtain the total cross section. Take the limit $m_\mu \to 0$. What is the problem with very high center of mass energies?

Hausaufgabe 17 Compton Scattering and the Klein-Nishima formula

In QED the electromagnetic field couples via the conserved vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$, $\partial_\mu j^\mu = 0$.
For matrix elements in Fourier space $\mathcal{M}^\mu(k) = \int d^4x\ e^{ikx}\langle f|j^\mu(x)|i\rangle$ this implies $k_\mu \mathcal{M}^\mu(k) = 0$ (known as the Ward identity).

a) Show that in an amplitude with an external photon

$$\sum_{\text{pols}} |\mathcal{M}(k)|^2 \equiv \sum_{\text{pols}} \left|\mathcal{M}^\mu(k)\epsilon^\star_\mu(k)\right|^2 = \mathcal{M}^\mu(k)\mathcal{M}^{\star\nu}(k)\sum_{\text{pols}} \epsilon^\star_\mu(k)\epsilon_\nu(k) \tag{7}$$

the sum over the physical polarizations of the photons can be replaced by $-g^{\mu\nu}$. For simplicity you may assume $k^\mu = (k,0,0,k)$ and $\epsilon_1^\mu = (0,1,0,0)$, $\epsilon_2^\mu = (0,0,1,0)$.

[**2 Punkte**]

b) Draw the two Feynman diagrams for $e^-(p)\gamma(k) \to e^-(p')\gamma(k')$ and write down the matrix element. Prove

$$(\not{p} + m)\gamma^\mu u(p) = 2p^\mu u(p) \tag{8}$$

in order to show that

$$i\mathcal{M} = -ie^2\epsilon^\star_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2pk'}\right]u(p). \tag{9}$$

[**2 Punkte**]

c) Calculate the squared matrix element, average over the initial and sum over the final state polarizations. You should obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(k)|^2 = \frac{e^4}{4}g_{\mu\rho}g_{\nu\sigma}\text{tr}\left\{(\not{p}' + m)\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2pk'}\right]\right.$$
$$\left.\times(\not{p} + m)\left[\frac{\gamma^\sigma \not{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2pk} + \frac{\gamma^\rho \not{k}'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2pk'}\right]\right\} \tag{10}$$

$$\equiv \frac{e^4}{4}\left[\frac{(i)}{(2pk)^2} + \frac{(ii)}{(2pk)(2pk')} + \frac{(iii)}{(2pk)(2pk')} + \frac{(iv)}{(2pk')^2}\right]. \tag{11}$$

[**2 Punkte**]

d) Prove the following relations:

$$\gamma_\mu\gamma^\mu = 4,\ \gamma_\mu\gamma^\sigma\gamma^\mu = -2\gamma^\sigma,\ \gamma_\mu\gamma^\sigma\gamma^\rho\gamma^\mu = 4g^{\sigma\rho},$$
$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu = -2\gamma^\alpha\gamma^\beta\gamma^\gamma. \tag{12}$$

[**3 Punkte**]

e) Show that evaluating the four traces yield (you obtain (iv) from (i) by interchanging k, k'):

$$\begin{aligned} (i):&\ \ 16\left[2m^4 + m^2(s-m^2) - 1/2(s-m^2)(u-m^2)\right] \\ (ii)=(iii):&-8\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right] \\ (iv):&\ \ 16\left[2m^4 + m^2(u-m^2) - 1/2(s-m^2)(u-m^2)\right] \end{aligned} \tag{13}$$

[**5 Punkte**]

f) Rewrite s and u in terms of $(p-k)$ and $(p-k')$ to arrive at

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(k)|^2 = 2e^4\left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right]. \tag{14}$$

[**1 Punkte**]

g) Take the initial electron to be at rest and parametrize the momenta in the following way:

$$p = (m, \vec{0}),\ k = (\omega, 0, 0, \omega),\ k' = (\omega', \omega'\sin\theta, 0, \omega'\cos\theta),\ p' = (E', \vec{p}'). \tag{15}$$

Use $m^2 = (p+k+k')^2$ to show that

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}. \tag{16}$$

[**2 Punkte**]

h) Show that the two-body phase space can be written as

$$\int d\Pi_2 = \int \frac{d\cos\theta}{2\pi}\frac{\omega'}{4E'}\frac{1}{\left|1 + \frac{\omega' - \omega\cos\theta}{E}\right|} = \frac{1}{8\pi}\int d\cos\theta \frac{(\omega')^2}{\omega m}. \tag{17}$$

[**3 Punkte**]

i) With this at hand, derive the Klein-Nishima formula

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{18}$$

[**2 Punkte**]

j) Take the limit $\omega \to 0$ to show that $\omega'/\omega \to 1$. Further compute the differential and the total cross section in this limit (you should recover the Thompson cross section for scattering classical electromagnetic radiation off an electron).

[**1 Punkte**]