Physikalisches Institut, Albert-Ludwigs-Universität Freiburg

Übungen zur Vorlesung

Theoretische Teilchenphysik

SS 08

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Compton Scattering and the Klein-Nishima formula

In QED the electromagnetic field couples via the conserved vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$, $\partial_\mu j^\mu = 0$.
For matrix elements in Fourier space $\mathcal{M}^\mu(k) = \int d^4x\ e^{ikx}\langle f|j^\mu(x)|i\rangle$ this implies $k_\mu \mathcal{M}^\mu(k) = 0$ (known as the Ward identity).

a) Show that in an amplitude with an external photon

$$\sum_{\text{pols}} |\mathcal{M}(k)|^2 \equiv \sum_{\text{pols}} \left|\mathcal{M}^\mu(k)\epsilon^\star_\mu(k)\right|^2 = \mathcal{M}^\mu(k)\mathcal{M}^{\star\nu}(k)\sum_{\text{pols}}\epsilon^\star_\mu(k)\epsilon_\nu(k) \tag{1}$$

the sum over the physical polarizations of the photons can be replaced by $-g^{\mu\nu}$. For simplicity you may assume $k^\mu = (k,0,0,k)$ and $\epsilon_1^\mu = (0,1,0,0)$, $\epsilon_2^\mu = (0,0,1,0)$.

b) Draw the two Feynman diagrams for $e^-(p)\gamma(k) \to e^-(p')\gamma(k')$ and write down the matrix element. Prove

$$(\not{p} + m)\gamma^\mu u(p) = 2p^\mu u(p) \tag{2}$$

in order to show that

$$i\mathcal{M} = -ie^2\epsilon^\star_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2pk'}\right]u(p). \tag{3}$$

c) Calculate the squared matrix element, average over the initial and sum over the final state polarizations. You should obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4}g_{\mu\rho}g_{\nu\sigma}\text{tr}\left\{(\not{p}' + m)\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2pk'}\right]\right.$$
$$\left.\times(\not{p} + m)\left[\frac{\gamma^\sigma\not{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2pk} + \frac{\gamma^\rho\not{k}'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2pk'}\right]\right\} \tag{4}$$
$$\equiv \frac{e^4}{4}\left[\frac{(i)}{(2pk)^2} + \frac{(ii)}{(2pk)(2pk')} + \frac{(iii)}{(2pk)(2pk')} + \frac{(iv)}{(2pk')^2}\right]. \tag{5}$$

d) Prove the following relations:

$$\gamma_\mu\gamma^\mu = 4,\ \gamma_\mu\gamma^\sigma\gamma^\mu = -2\gamma^\sigma,\ \gamma_\mu\gamma^\sigma\gamma^\rho\gamma^\mu = 4g^{\sigma\rho},$$
$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu = -2\gamma^\gamma\gamma^\beta\gamma^\alpha. \tag{6}$$

e) Show that evaluating the four traces yields (you obtain (iv) from (i) by interchanging $k \leftrightarrow k'$):

$$\begin{aligned}(i) :& 16\left[2m^4 + m^2(s-m^2) - 1/2(s-m^2)(u-m^2)\right] \\ (ii) = (iii) :& -8\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right] \\ (iv) :& 16\left[2m^4 + m^2(u-m^2) - 1/2(s-m^2)(u-m^2)\right]\end{aligned} \tag{7}$$

f) Rewrite s and u in terms of $(p \cdot k)$ and $(p \cdot k')$ to arrive at

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right]. \tag{8}$$

g) Take the initial electron to be at rest and parametrize the momenta in the following way:

$$p = (m, \vec{0}),\ k = (\omega, 0, 0, \omega),\ k' = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta),\ p' = (E', \vec{p}'). \tag{9}$$

Use $m^2 = (p + k - k')^2$ to show that

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}. \tag{10}$$

h) Show that the two-body phase space can be written as

$$\int d\Pi_2 = \int \frac{d\cos\theta}{2\pi} \frac{\omega'}{4E'} \frac{1}{\left|1 + \frac{\omega' - \omega\cos\theta}{E'}\right|} = \frac{1}{8\pi}\int d\cos\theta \frac{(\omega')^2}{\omega m}. \tag{11}$$

i) With this at hand, derive the Klein-Nishina formula

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{12}$$

j) Take the limit $\omega \to 0$ to show that $\omega'/\omega \to 1$. Further compute the differential and the total cross section in this limit (you should recover the Thomson cross section for scattering classical electromagnetic radiation off an electron).