Übungen zur Vorlesung

Theoretische Teilchenphysik

SS 08

Prof. Dr. J. Reuter
(Hochhaus/803), 8407

Blatt 7
05.06.08

Präsenz-Aufgabe 15 The Ward-Takahashi identity

The Ward-Takahashi identity implies that if $\mathcal{M}(k) = \epsilon_\mu(k)\mathcal{M}^\mu(k)$ is the amplitude for any QED process involving an external photon with momentum k, then this amplitude vanishes if we replace $\epsilon_\mu(k)$ by k_μ:

$$k_\mu \mathcal{M}^\mu(k) \;=\; 0. \tag{1}$$

To prove this relation, it is useful to prove a more general identity for QED correlation functions. $\mathcal{M}^\mu(k)$ shall denote from now on the Fourier-transformed correlation function, in which external momenta are not necessarily on-shell. In order to prove the Ward-Takahashi identity, we consider a general QED correlation function with an arbitrary number of external photons and electrons and then insert an additional external photon with momentum k at every possible insertion point and sum over all contributions. In QED an external photon line is either attached to an electron line that $a)$ runs between external points or $b)$ to an electron loop. We will treat both cases separately.

a) First, attach an additional photon to the electron line between the i-th and $i+1$-th photon vertex:

$$\rightarrow$$

On the left-hand side the electron propagators have momenta p, $p_1 = p + q_1$, ..., $p' = p_{n-1} + q_n$. After inserting the additional photon line the electron propagators to the left of the photon have their momenta increased by k.
Now, let us evaluate the value of the diagram on the right-hand side with k replacing the corresponding polarization vector $\epsilon_\mu(k)$.
Show that the product of the new vertex multiplied by the adjacent electron propagators is given by

$$\frac{i}{\not p_i + \not k - m}\,(-ie\not k)\,\frac{i}{\not p_i - m} = e\left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not k - m}\right). \tag{2}$$

Evaluate the diagrams with the photon inserted at position i and $i-1$. Comparing the two expression should show you that summing over all possible insertion points yields a telescopic sum. The result is the difference of only the two diagrams where in the first the outgoing momentum is increased by k and in the second one the incoming momentum is $p+k$ whereas all other momenta remain unchanged.

$$\sum_{\text{insertation points}} k_\mu \times \left(\qquad \right)$$

To obtain the contribution to the S-matrix element, insert the inverse propagators for the external electrons

$$(\not{p}' + \not{k} - m)\,(\not{p} - m)$$

and put the electron momenta on-shell. You should obtain eq.(1).

b) Now consider the case where the additional photon is attached to an electron loop with n photon vertices. Insert the photon between the first and the second vertex, as well as in between the third and fourth. Summing over all possible insertions, in order to obtain

$$-e\int \frac{d^4p_1}{(2\pi)^4}\text{tr}\left[\left(\frac{i}{\not{p}_n - m}\right)\gamma^{\lambda_n}\left(\frac{i}{\not{p}_{n-1} - m}\right)\gamma^{\lambda_{n-1}}\cdots\left(\frac{i}{\not{p}_1 - m}\right)\gamma^{\lambda_1}\right.$$
$$\left.-\left(\frac{i}{\not{p}_n + \not{k} - m}\right)\gamma^{\lambda_n}\left(\frac{i}{\not{p}_{n-1} + \not{k} - m}\right)\gamma^{\lambda_{n-1}}\cdots\left(\frac{i}{\not{p}_1 + \not{k} - m}\right)\gamma^{\lambda_1}\right].$$

This integral vanishes - why is that?

c) Very often the terminus *Ward-Takahashi identity* is used to mean the special case of

$$-ik_\mu\Gamma^\mu(p+k,p) = S_F^{-1}(p+k) - S_F^{-1}(p), \tag{3}$$

where Γ denotes the full QED vertex factor and S is the fermion propagator. How does this relation follow from the above theorem?

Hausaufgabe 10 Linear sigma model

Investigate the properties of a spontaneously broken *global* symmetry in a so-called *linear sigma model*. The Lagrangian involves a set of N real scalar fields $\phi^i(x)$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right], \tag{4}$$

with an implicit summation over i. It is invariant under symmetry transformations

$$\phi^i \to O^{ij}\phi^j \quad \forall \quad N-\text{dim othorgonal matrices } O \tag{5}$$

a) Minimize the potential. Note that the minimalization only fixes the length of the constant vector ϕ_0^i, whereas the direction remains arbitrary. Choose it's direction to be the N-th direction.

b) Define a set of fields shifted by the vacuum expectation value v

$$\phi(x) = \big(\pi^1(x), ..., \pi^{N-1}(x), v + \sigma(x)\big) \tag{6}$$

and rewrite the Lagrangian () in terms of the new fields.

c) Interpret the field content and derive the Feynman rules.

d) Derive the cross section for $2 \to 2$ Nambu-Goldstone boson scattering.