Physikalisches Institut, Albert-Ludwigs-Universität Freiburg

Übungen zur Vorlesung

Theoretische Teilchenphysik

SS 08

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Präsenz-Aufgabe 1 Technicalities for path integrals

Show that a Gaussian is an incarnation of the Delta function, i.e.

$$\lim_{\epsilon\to 0}\ \delta_\epsilon \ \equiv\ \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon c}}\exp\left[-\frac{x^2}{\epsilon c}\right]\ =\ \delta(x) \tag{1}$$

with an arbitrary constant c. This provides the normalization of the quantum mechanical path integral.

Präsenz-Aufgabe 2 Gaussian integrals

Show that

a) for any real symmetric matrix A:

$$\int\frac{d^n x}{(2\pi)^{n/2}}\exp\left[-\frac{1}{2}x^T A x\right] = \frac{1}{\sqrt{\det A}}.$$

b) over $\mathbb{C}$ with $z = x+iy$, $z^\star = x - iy$:

$$\int\frac{dz dz^\star}{\sqrt{2\pi i}\sqrt{2\pi i}}\exp\left[-a z^\star z\right] = \frac{1}{a}.$$

c) for $A = A^\dagger$:

$$\int\frac{d^n z d^n z^\star}{(2\pi i)^{n/2}(2\pi i)^{n/2}}\exp\left[-z^\dagger A z\right] = \frac{1}{\det A}.$$

d) Show the behavior of the Gaussian under translations:

$$\int\frac{d^n x}{(2\pi)^{n/2}}\exp\left[-\frac{1}{2}x^T A x - b^T x\right] = \frac{\exp\left[\frac{1}{2}b^T A^{-1} b\right]}{\sqrt{\det A}}.$$

$$\int\frac{d^n z d^n z^\star}{(2\pi i)^{n/2}(2\pi i)^{n/2}}\exp\left[-z^\dagger A z + u^\dagger z + z^\dagger u\right] = \frac{\exp\left[u^\dagger A^{-1} u\right]}{\det A}.$$

e) What about infinite dimensional function spaces with the inner product

$$(\phi, A\phi) = \int dx\, dy\, \phi^\star(x) A(x,y)\phi(y) \quad \overset{A(x-y)=A(x)\delta(x-y)}{\longrightarrow} \quad \int dx\, \phi^\star(x) A(x)\phi(x)?$$

Präsenz-Aufgabe 3 Functional derivative

Define the functional derivative as

$$\frac{\delta F[f](x)}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ F[f(x) + \epsilon \delta(x-y)] - F[f(x)] \right\}.$$

Calculate the functional derivatives $\frac{\delta F[f](x)}{\delta f(y)}$ according to this definition for the following functionals:

a)

$$F[f] = \int dx\, f(x),$$

b)

$$F[f] = \int dz\, G(x,z) f(z),$$

c)

$$F[f] = f(x), \quad \text{i.e. } F \equiv 1 \text{ or } F \equiv \text{id},$$

d)

$$F[f] = \exp\left[\int dx\, f(x)\right].$$

Hausaufgabe 1 Physics of a massive vector boson and collinear photon radiation

Add to QED a massive photon field B_μ of mass M, which couples to electrons via

$$\Delta H = \int d^3x \, \left(g \bar{\psi} \gamma^\mu \psi B_\mu \right).$$

A massive photon in the initial or final state has three physical polarizations, corresponding to three space-like unit vectors in the boson's rest frame. These can be characterized invariantly, in terms of the boson's 4-momentum k_μ, as three vectors $\epsilon_\mu^{(i)}$ satisfying

$$\epsilon^{(i)} \epsilon^{(j)} = -\delta^{ij}, \quad k^\mu \epsilon_\mu^{(i)} = 0.$$

The four vectors $(k_\mu/M, \epsilon_\mu^{(i)})$ form a complete orthonormal basis. Since B_μ couples to the conserved current $\bar{\psi}\gamma^\mu\psi$, the Ward identity implies that k_μ dotted into the amplitude for B production yields zero; thus we can replace:

$$\sum_i \epsilon_\mu^{(i)} \epsilon_\nu^{(i)} \to -g_{\mu\nu}.$$

This gives a generalization to massive bosons of the Feynman trick for photon polarizations and simplifies the calculation of B production cross sections. (Warning: This trick does not work (so simply) for non-Abelian gauge fields.) In the following calculations a vanishing electron mass shall be assumed.

a) Compute the cross section for the process $e^+e^- \to B$. Compute the lifetime of the B, assuming that it decays only to electrons. Verify the relation

$$\sigma(e^+e^- \to B) = \frac{12\pi^2}{M}\Gamma(e^+e^- \to B)\delta(M^2 - s).$$

b) Compute the differential cross section, in the center-of-mass system, for the process $e^+e^- \to \gamma + B$. (This calculation goes over almost unchanged to the realistic process $e^+e^- \to \gamma + Z$; this allows one to measure the number of Z decays into unobserved final states, which in turn is proportional to the number of neutrino species.)

c) Notice that the cross section of part b) diverges as $\theta \to 0$ or π. Let us analyze the region near $\theta = 0$. In this region, the dominant contribution comes from the t-channel diagram and corresponds intuitively to the emission of a photon from the electron line before e^+e^- annihilation into a B. Let us rearrange the formula in such a way to support this interpretation. First, note that the divergence as $\theta \to 0$ is cut off by the electron mass: Let the electron momentum be $p^\mu = (E, 0, 0, k)$, with $k = \sqrt{E^2 - m^2}$, and let the photon momentum be $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$. Show that the denominator of the propagator never becomes smaller than $\mathcal{O}(m^2/s)$: Now integrate the cross section of part b) over forward angles, cutting off the θ integral at $\theta^2 \approx \mathcal{O}(m^2/s)$ and keeping only the leading logarithmic term, proportional to $\log(s/m^2)$. Show that, in this approximation, the cross section for forward photon emission can be written as

$$\sigma(e^+e^- \to \gamma + B) \approx \int dx\, f(x)\, \sigma(e^+e^- \to B \text{ at } E_{cm}^2 = (1-x)s),$$

where the annihilation cross section is evaluated for the collision of a positron of energy E with an electron of energy $(1-x)E$, and the function $f(x)$, the *Weizsäcker-Williams distribution function*, is given by

$$f(x) = \frac{\alpha}{2\pi}\frac{1 + (1-x)^2}{x}\log\left(\frac{s}{m^2}\right).$$

This function arises universally in processes in which a photon is emitted collinearly from an electron line, independent of the subsequent dynamics.