Solutions to Selected Problems on Bunch Compressors

1. **Derive equations (3) through (6)**

Equation (2) leads to the expression for the final longitudinal beta parameter, in terms of the initial Twiss parameters, the phase slip R_{56} and the RF parameter R_{65}:

$$\beta_1 = \beta_0 \left(1 + R_{56} R_{65}\right)^2 - 2\alpha_0 R_{56} \left(1 + R_{56} R_{65}\right) + \gamma_0 R_{56}^2$$

Minimizing β_1 with respect to R_{56}:

$$\frac{\partial \beta_1}{\partial R_{56}} = 0$$

gives:

$$R_{56} = \frac{\alpha_0 - R_{65}\beta_0}{\gamma_0 - 2R_{65}\alpha_0 + R_{65}^2 \beta_0}$$

Substituting into the above expression for β_1 gives:

$$\breve{\beta}_1 = \frac{1}{\gamma_0 - 2R_{65}\alpha_0 + R_{65}^2 \beta_0}$$

Thus we have equations (3) and (4).

In the special case $\alpha_0 = 0$, we have $\gamma_0 = 1/\beta_0$. Since $\sigma_z = \sqrt{\beta\varepsilon}$ and the longitudinal emittance ε is preserved, we have:

$$\left(\frac{\sigma_{zf}}{\sigma_{zi}}\right)_{\min} = \sqrt{\frac{\breve{\beta}_1}{\beta_0}} = \frac{1}{\sqrt{\beta_0\gamma_0 + R_{65}^2\beta_0^2}} = \frac{1}{\sqrt{1 + R_{65}^2\beta_0^2}}$$

We also have:

$$\left(\frac{\sigma_{zi}}{\sigma_{\delta i}}\right)^2 = \frac{\beta_0\varepsilon}{\gamma_0\varepsilon} = \beta_0^2$$

and hence we find equation (5):

$$\left(\frac{\sigma_{zf}}{\sigma_{zi}}\right)_{\min} = \frac{1}{\sqrt{1 + a^2}} \qquad a = \frac{\sigma_{zi}}{\sigma_{\delta i}} R_{65}$$

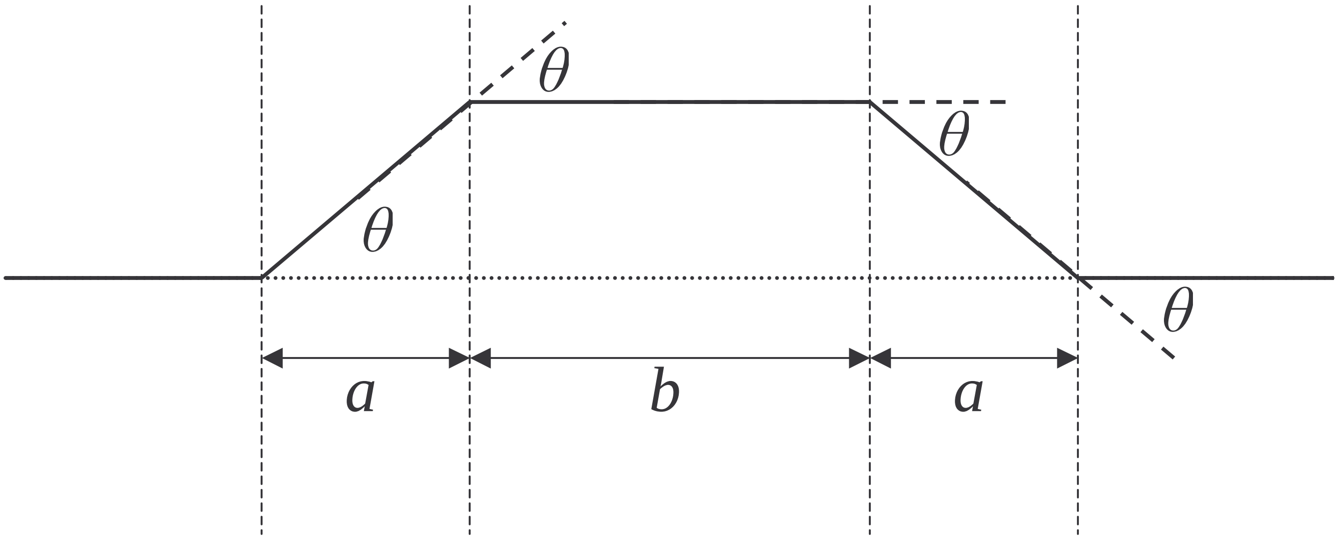
Finally, the above expression for R_{56} is:

$$R_{56} = \frac{\alpha_0 - R_{65}\beta_0}{\gamma_0 - 2R_{65}\alpha_0 + R_{65}^2\beta_0} = -\frac{R_{65}\beta_0}{\gamma_0 + R_{65}^2\beta_0} = -\frac{R_{65}\beta_0^2}{1 + R_{65}^2\beta_0^2} = -\frac{a^2}{1+a^2}\cdot\frac{1}{R_{65}}$$

which gives equation (6).

2. Find the change in path length with respect to the relative energy error, for a chicane constructed from four symmetrically placed short dipoles.

The chicane is shown in the following diagram:



The dipoles bend particles in the beam through angle θ. If a particle at the reference energy is bent through an angle θ_0, then a particle with relative energy error δ is bent through an angle:

$$\theta = \frac{\theta_0}{1+\delta}$$

The path length from the first to final dipole is given by:

$$s = \frac{2a}{\cos(\theta)} + b = 2a\left[\cos\left(\frac{\theta_0}{1+\delta}\right)\right]^{-1} + b$$

Assuming that the bending angle is small, we can make the approximation:

$$\left[\cos\left(\frac{\theta_0}{1+\delta}\right)\right]^{-1} \approx 1+\frac{1}{2}\left(\frac{\theta_0}{1+\delta}\right)^2$$

and hence:

$$s \approx 2a + a\left(\frac{\theta_0}{1+\delta}\right)^2 + b$$

R_{56} is the first order change in path length with respect to the energy error, at the design energy:

$$R_{56} = \left.\frac{ds}{d\delta}\right|_{\delta=0} \approx -2a\left.\frac{\theta_0^2}{(1+\delta)^3}\right|_{\delta=0} = -2a\theta_0^2$$

Therefore, we can write the change in path length with respect to the relative energy error as:

$$\Delta s = s(\delta) - s(\delta=0) \approx a\left(\frac{\theta_0}{1+\delta}\right)^2 - a\theta_0^2 = \tfrac{1}{2}R_{56}\left(1-\frac{1}{(1+\delta)^2}\right)$$

Carrying out a Taylor expansion in the relative energy error about $\delta = 0$ gives:

$$(1+\delta)^{-2} = 1 - 2\delta + 3\delta^2 - 4\delta^3 + \cdots$$
$$\Delta s \approx R_{56}\delta - \tfrac{3}{2}R_{56}\delta^2 + 2R_{56}\delta^3 - \cdots$$

and hence:

$$T_{566} \approx -\tfrac{3}{2}R_{56}$$
$$U_{5666} \approx 2R_{56}$$

etc.

Now consider placing one or more quadrupoles in the above system. Assuming the quadrupoles are placed so that the design beam goes through the center of each quadrupole, the quadrupoles provide a bending:

$$\theta_{quad} = k_1 l \eta \delta$$

where η is the dispersion at the quadrupole location, and $k_1 l$ is the integrated focusing strength of the quadrupole. The dipoles in the chicane generate non-zero dispersion, and the quadrupoles will therefore provide additional bending for particles with some energy error. (Note also that the bending from the quadrupoles *increases* linearly with the energy error, rather than *decreasing* as in the case of the dipoles). The geometry of the system is therefore changed, and the dependence of path length of energy error will take a different form to that given in the above expression.

3. Using the linear model, calculate for the TESLA and NLC Stage I bunch compressors, the required RF voltage and the required phase slip R_{56}.

Using the data in Table 1, and equations (5) and (6) we find:

	TESLA	NLC Stage I
σ_{zf}/σ_{zi}	0.05	0.1
$a=\sqrt{(\sigma_{zi}/\sigma_{zf})^2-1}$	20	10
$R_{65}=a\sigma_{\delta i}/\sigma_{zi}$	4.33 m^{-1}	2.00 m^{-1}
$V_{RF}=\frac{E_0}{e}\frac{c}{2\pi f_{RF}}R_{56}$	795 MV	132 MV
$R_{56}\approx\frac{1}{R_{65}}$	0.23 m	0.5 m

4. Calculate the nonlinear phase slip compensation for the TESLA and NLC Stage I bunch compressors.

Neglecting any change in the longitudinal emittance from the energy change of the beam, the value of R_{65} is determined from equation (5) by the required compression ratio. Using the RF phase for the nonlinear phase slip compensation, we have:

$$R_{65}=\theta\sin(\phi_{RF})k_{RF}$$

where $\theta=\frac{eV_{RF}}{E_0}$. Equation (7) gives us:

$$\cos(\phi_{RF})=\frac{\sqrt{1+8(1+2r)r\theta^2}-1}{2(1+2r)\theta}$$

where $r = \frac{T_{566}}{R_{56}}$. For a simple chicane, $r \approx -1.5$. The above equations may be solved numerically for θ and ϕ_{RF}. We find:

	TESLA	NLC Stage I
V_{RF}	895 MV	135 MV
ϕ_{RF}	-117°	-101°

5. Derive equations (8) and (9).

If there is no acceleration, there is no damping, so the only change in action of a particle comes from the quantum excitation. For the emission of a single photon of energy *u* we have:

$$\Delta J = \frac{1}{2}\left(\frac{u}{E_0}\right)^2 \mathcal{H}$$

Hence the rate of change of the action may be written in terms of the mean rate of photon emission *N*:

$$\frac{dJ}{ds} = \frac{1}{2cE_0^2} N\langle u^2 \rangle \mathcal{H} = \frac{1}{2cE_0^2} 2C_q \gamma^2 E_0 \frac{P_\gamma}{\rho} \mathcal{H} = \frac{1}{2cE_0^2} 2C_q \gamma^2 E_0 \frac{cC_\gamma}{2\pi} E_0^4 \frac{\mathcal{H}}{\rho^3}$$

Tidying up, we have:

$$\Delta J = C_q \gamma^2 E_0^3 \frac{C_\gamma}{2\pi} \int \frac{\mathcal{H}}{\rho^3} \mathrm{d}s = C_q \gamma^2 E_0^3 \frac{C_\gamma}{2\pi} I_5$$

Now we use:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 \left(m_e c^2\right)^4} \qquad r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}$$

to write:

$$\Delta J = \tfrac{2}{3} C_q r_e \gamma^5 I_5$$

But the emittance of the beam is just the mean action of all particles in the beam, and thus (including a factor γ to normalize the emittance) we arrive at:

$$\Delta(\gamma\varepsilon) = \tfrac{2}{3} C_q r_e \gamma^6 I_5$$

The change in energy spread of the beam along a beamline (without acceleration) is:

$$\Delta\sigma_\delta^2 = \frac{1}{cE_0^2}\int N\langle u^2\rangle \mathrm{d}s = \frac{1}{cE_0^2} 2C_q\gamma^2 E_0 \int \frac{P_\gamma}{\rho}\mathrm{d}s = \frac{1}{cE_0^2} 2C_q\gamma^2 E_0 \frac{cC_\gamma}{2\pi} E_0^4 \int \frac{1}{\rho^3}\mathrm{d}s$$

This becomes:

$$\Delta\sigma_\delta^2 = 2C_q\gamma^2 E_0^3 \frac{C_\gamma}{2\pi}\int \frac{1}{\rho^3}\mathrm{d}s = 2C_q\gamma^2 E_0^3 \frac{C_\gamma}{2\pi} I_3$$

Using the above expression for C_γ, this becomes:

$$\Delta\sigma_\delta^2 = \tfrac{4}{3} C_q r_e \gamma^5 I_3$$