Solutions to Selected Problems on Damping Rings

1. **Show that:** $\oint \frac{P_\gamma}{\rho} ds = \frac{I_3}{I_2} \oint P_\gamma ds$

Starting from Equation (7), we proceed as follows:

$$P_\gamma = \frac{cC_\gamma}{2\pi} \frac{E_0^4}{\rho^2}$$
$$\oint P_\gamma ds = \frac{cC_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} ds = \frac{cC_\gamma}{2\pi} E_0^4 I_2$$
$$\oint \frac{P_\gamma}{\rho} ds = \frac{cC_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^3} ds = \frac{cC_\gamma}{2\pi} E_0^4 I_3$$

and therefore:

$$\oint \frac{P_\gamma}{\rho} ds = \frac{I_3}{I_2} \oint P_\gamma ds$$

2. **Show that the betatron action induced by a particle emitting a photon of energy *u* is given by:**

$$\Delta J = \frac{1}{2} \left(\frac{u}{E_0} \right)^2 \mathcal{H}$$

Starting from the definition for the action:

$$2J = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

with a change in energy $\delta = u/E_0$ inducing a change in co-ordinates:

$$x_1 = x_0 + \eta\delta$$
$$x_1' = x_0' + \eta'\delta$$

we find a change in the action given by:

$$2J_1 = 2J_0 + 2\delta\left[\gamma x_0 \eta + \alpha\left(x_0\eta' + x_0'\eta\right) + \beta x_0'\eta'\right] + \delta^2\left(\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2\right)$$

Note that the second term, linear in δ, is also linear in the co-ordinates, and vanishes if we average over the betatron phase angle (the conjugate variable to the action). This means that the mean change in the action is given by:

$$\langle \Delta J \rangle = \tfrac{1}{2}\delta^2\left(\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2\right)$$

as advertised.

3. **Show that:** $\dfrac{1}{2E_0^2 C_0}\oint N\langle u^2\rangle \mathcal{H} \mathrm{d}s = 2C_q \dfrac{\gamma^2}{\mathcal{J}\tau}\dfrac{I_5}{I_2}$

Using:

$$N\langle u^2\rangle = 2C_q\gamma^2 E_0 \frac{P_\gamma}{\rho}$$
$$P_\gamma = \frac{cC_\gamma}{2\pi}\frac{E_0^4}{\rho^2}$$

we find:

$$\frac{1}{2E_0^2 C_0}\oint N\langle u^2\rangle \mathcal{H} \mathrm{d}s = \frac{1}{E_0 C_0} C_q \gamma^2 \frac{cC_\gamma}{2\pi} E_0^4 \oint \frac{\mathcal{H}}{\rho^3}\mathrm{d}s = \frac{1}{E_0 C_0} C_q \gamma^2 \frac{cC_\gamma}{2\pi} E_0^4 I_5$$

Now using:

$$\oint P_\gamma \mathrm{d}s = \frac{cC_\gamma}{2\pi} E_0^4 I_2 = cU_0$$

we find:

$$\frac{1}{2E_0^2 C_0}\oint N\langle u^2\rangle \mathcal{H} \mathrm{d}s = \frac{cU_0}{E_0 C_0} C_q \gamma^2 \frac{I_5}{I_2}$$

Finally, we use:

$$\frac{cU_0}{E_0 C_0} = \frac{U_0}{E_0 T_0} = \frac{2}{\mathcal{J}\tau}$$

to arrive at:

$$\frac{1}{2E_0^2 C_0}\oint N\langle u^2\rangle \mathcal{H} \mathrm{d}s = 2\frac{C_q \gamma^2}{\mathcal{J}\tau}\frac{I_5}{I_2}$$

4. Show that for $1/\rho = 0$, $\mathcal{H}$ is a constant.

If there is no bending between *s*=0 and *s*, then the dispersion evolves as:

$$\vec{\eta}(s) = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_s = \mathbf{M} \cdot \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_0 = \mathbf{M} \cdot \vec{\eta}(0)$$

where **M** is the transfer matrix from 0 to *s*. In terms of the matrix **A** (constructed from the Twiss functions), $\mathcal{H}$ can be written as:

$$\mathcal{H} = \vec{\eta}^{\mathrm{T}} \cdot \mathbf{A} \cdot \vec{\eta}$$

and **A** evolves as:

$$\mathbf{A}(s) = \mathbf{M} \cdot \mathbf{A}(0) \cdot \mathbf{M}^{\mathrm{T}}$$

Thus, we can write:

$$\begin{aligned} \mathcal{H}(s) &= \vec{\eta}^{\mathrm{T}}(s) \cdot \mathbf{A}(s) \cdot \vec{\eta}(s) \\ &= \vec{\eta}^{\mathrm{T}}(0) \cdot \mathbf{M}^{\mathrm{T}} \cdot \mathbf{M} \cdot \mathbf{A}(0) \cdot \mathbf{M}^{\mathrm{T}} \cdot \mathbf{M} \cdot \vec{\eta}(0) \end{aligned}$$

Since **M** must be a 2×2 symplectic matrix, we can write:

$$\mathbf{M}^{\mathrm{T}} \cdot \mathbf{M} = \mathbf{I}$$

and hence,

$$\mathcal{H}(s) = \vec{\eta}^{\mathrm{T}}(0) \cdot \mathbf{A}(0) \cdot \vec{\eta}(0) = \mathcal{H}(0)$$

5. Use the parameters given in Table 1 to calculate the synchrotron radiation integrals, damping times, equilibrium energy spread and natural emittance for the NLC and TESLA damping rings, assuming only arc cells in the lattices (i.e. wigglers turned off).

Note: where needed, we can calculate the evolution of the dispersion and the mean H-function in the dipoles using the expressions on pages 19 and 20. The TESLA lattice has 108 arc cells, and the dipole bending radius is 85.92 m. The NLC lattice has 36 arc cells, and the dipole bending radius is 5.5 m.

Using expressions for the synchrotron radiation integrals given on page 18, we find the following results:

	TESLA	NLC MDR
I_1 (arcs)	2.153 m	0.0922 m
I_2 (arcs)	0.1317 m^{-1}	1.140 m^{-1}
I_3 (arcs)	1.532×10^{-3} m^{-2}	0.2080 m^{-2}
I_4 (arcs)	2.92×10^{-4} m^{-1}	-0.195 m^{-1}
I_5 (arcs)	1.32×10^{-5} m^{-1}	2.05×10^{-4} m^{-1}

Using these values in the expressions on page 17, we find the following parameters:

	TESLA	NLC MDR
α_p	1.27×10^{-4}	3.07×10^{-4}
U_0	1.16 MeV	0.247 MeV
σ_δ	4.62×10^{-4}	7.57×10^{-4}
$\mathcal{J}_\varepsilon$	2.0	1.83
$\mathcal{J}_x$	1.0	1.17
$\mathcal{J}\tau$	489 ms	16.0 ms
$\gamma\varepsilon_0$	36.0 μm	3.43 μm

Note that the lattices with the wigglers turned off have much longer damping times than necessary (28 ms for TESLA and 5 ms for NLC), and larger emittances than specified (8 μm for TESLA and 3 μm for NLC).

6. Using the wiggler parameters given in Table 2, calculate the synchrotron radiation integrals for the NLC and TESLA damping rings.

Using the approximate expressions given on page 24, we find:

	TESLA	NLC MDR
I_1 (wiggler)	-8.87×10^{-3} m	-4.52×10^{-3} m
I_2 (wiggler)	2.19 m^{-1}	2.46 m^{-1}
I_3 (wiggler)	0.179 m^{-2}	0.679 m^{-2}
I_4 (wiggler)	-6.15×10^{-5} m^{-1}	-3.60×10^{-4} m^{-1}
I_5 (wiggler)	1.47×10^{-5} m^{-1}	1.59×10^{-4} m^{-1}

The total values for the synchrotron radiation integrals are found simply by adding the values for the arcs to the values for the wigglers. We then find the final values for the parameters:

	TESLA	NLC MDR
α_p	1.26×10^{-4}	2.92×10^{-4}
U_0	20.4 MeV	0.779 MeV
σ_δ	12.0×10^{-4}	8.4×10^{-4}
$\mathcal{J}_\varepsilon$	2.0	1.95
$\mathcal{J}_x$	1.0	1.05
$\mathcal{J}\tau$	28 ms	5.0 ms
$\gamma\varepsilon_0$	4.3 μm	2.2 μm

Note:

- The wigglers make a negative contribution to the phase slip (I_1), but this is small compared to the contribution from the arcs, so the momentum compaction is not much affected by the wigglers.
- The wigglers make a dominant contribution to the energy loss per turn (I_2). This greatly enhances the damping rate, or (in other words) reduces the damping times.
- The wigglers make a strong contribution to the quantum excitation of the energy spread (I_3). However, the equilibrium energy spread is a balance between the quantum excitation and the radiation damping, and since the latter (I_2) has also greatly increased, the equilibrium energy spread is not increased as much as might be expected.
- The NLC dipoles have a strong quadrupole component, so the horizontal damping partition (I_4) is changed significantly from the nominal value of 1. This effect is compensated by the wiggler, however, which makes only a small contribution to I_4, while greatly increasing I_2.
- The wigglers make a significant contribution to the quantum excitation of betatron oscillations (I_5). However, the natural emittance is actually reduced, because the radiation damping (I_2) dominates.

7. (Chromaticity calculations)

8. Show that the RF acceptance of a storage ring is given by:

$$\delta_{RF}^2 = -\frac{4eV_{RF}}{E_0 T_0 \omega_{RF} \alpha_p}\left[\cos(\phi_s) + \left(\phi_s - \frac{\pi}{2}\right)\sin(\phi_s)\right]$$

Refer to Figure 7 on page 29. The phase space has two fixed points, a stable point (the center of the loop) at $\tau = 0$, and an unstable point (where the contours intersect) at $\tau = \tau_u$. A particle crossing the RF cavity with zero energy error and time $\tau = \tau_u$

with respect to the reference particle must see the same RF voltage as the reference particle, since neither perform synchrotron oscillations. If the synchronous phase is ϕ_s and the RF voltage is sin-like, then we must have:

$$\sin(\phi_s - \omega\tau_u) = \sin(\phi_s)$$

This equation has the non-trivial solution:

$$\omega\tau_u = 2\left(\phi_s - \frac{\pi}{2}\right)$$

We can now calculate the value of the Hamiltonian on the separatrix, by evaluating

$$H = -\tfrac{1}{2}\alpha_p\delta^2 - \frac{eV_{RF}}{E_0 T_0 \omega_{RF}}\left[\cos(\phi_s - \omega_{RF}\tau) - \sin(\phi_s)\omega_{RF}\tau\right]$$

at $(\delta = 0, \tau = \tau_u)$:

$$H_{sep} = \frac{eV_{RF}}{E_0 T_0 \omega_{RF}}\left[\cos(\phi_s) + (2\phi_s - \pi)\sin(\phi_s)\right]$$

The limit of the RF acceptance, δ_{RF}, also lies on the separatrix, but at $\tau = 0$. Thus, we have:

$$H_{sep} = -\frac{1}{2}\alpha_p\delta_{RF}^2 - \frac{eV_{RF}}{E_0 T_0 \omega_{RF}}\cos(\phi_s)$$

Combining the above two expressions for H_{sep} gives us:

$$\delta_{RF}^2 = -\frac{2eV_{RF}}{E_0 T_0 \omega_{RF}\alpha_p}\left[2\cos(\phi_s) + (2\phi_s - \pi)\sin(\phi_s)\right]$$

9. Calculate the RF voltage required to give an RF acceptance of 1.5% in the NLC and TESLA damping rings. Also calculate the synchrotron frequency, and the equilibrium bunch length.

We use the above expression for the RF acceptance. We also have the relationship between the RF voltage and synchronous phase:

$$U_0 = eV_{RF}\sin(\phi_s)$$

Putting these equations together, we have two equations, in two unknowns (the RF voltage and synchronous phase). Eliminating the RF voltage gives us:

$$\cot(\phi_s) + \left(\phi_s - \frac{\pi}{2}\right) = -\frac{E_0 T_0 \omega_{RF} \alpha_p \delta_{RF}^2}{4U_0}$$

With values of 500 MHz for the TESLA RF frequency, and 714 MHz for the NLC RF frequency, this equation may be solved numerically to find the synchronous phase. We then use this value to find the following parameters:

	TESLA	NLC MDR
ϕ_s	2.44	2.33
V_{RF}	31 MV	1.1 MV
ω_s	5.76×10^3 s^{-1}	22.1×10^3 s^{-1}
σ_z	7.9 mm	3.5 mm

10. Show that the fixed points of the betatron coupling Hamiltonian are given by equations (26).

First, we note that we are looking for the fixed points in the horizontal and vertical actions, not fixed points of the phase angles. The equations of motion, found from the Hamiltonian (25), are:

$$\frac{\mathrm{d}J_x}{\mathrm{d}s} = |\tilde{\kappa}|\sqrt{J_x J_y}\,\sin(\phi_x - \phi_y) \qquad \frac{\mathrm{d}\phi_x}{\mathrm{d}s} = \frac{2\pi}{C_0}\left[\nu_x + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_y}{J_x}}\cos(\phi_x - \phi_y)\right]$$

$$\frac{\mathrm{d}J_y}{\mathrm{d}s} = -|\tilde{\kappa}|\sqrt{J_x J_y}\,\sin(\phi_x - \phi_y) \qquad \frac{\mathrm{d}\phi_y}{\mathrm{d}s} = \frac{2\pi}{C_0}\left[\nu_y + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_x}{J_y}}\cos(\phi_x - \phi_y)\right]$$

At the fixed points, we have:

$$\frac{\mathrm{d}J_x}{\mathrm{d}s} = \frac{\mathrm{d}J_y}{\mathrm{d}s} = 0$$

and thus:

$$\phi_x(s) = \phi_y(s)$$

$$\therefore \qquad \frac{\mathrm{d}\phi_x}{\mathrm{d}s} = \frac{\mathrm{d}\phi_y}{\mathrm{d}s}$$

$$\therefore \qquad \nu_x + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_y}{J_x}} = \nu_y + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_x}{J_y}}$$

Defining:

$$J_0 = J_x + J_y$$

equations (26) follow.