Lecture 1: Problems

***Q1**: The earth's radius is approximately 6400 km. Pretend we can build a storage ring collider around the planet which sits in geostationary orbit. Assuming that the earth's magnetic field has a constant value of 0.5 Gauss (= 5 mT) around our machine, and that it is perfectly perpendicular to the equatorial plane, calculate (a) the centre of mass energy of the machine, and (b) the average power radiated per electron.*

Answer:

We can immediately calculate the beam energy from $B r = P / c \approx 3.34 P[\mathrm{GeV/c}]$:

$$
\begin{aligned}
P[\mathrm{GeV/c}] &\approx \frac{B[\mathrm{Tesla}]\, r\,[\mathrm{m}]}{3.34} \\
&\approx \frac{(5\times10^{-6})\times(6.4\times10^{6})}{3.34} \\
&\approx 9.6\ \mathrm{GeV/c}
\end{aligned}
$$

So the centre of mass energy is ~20 GeV. The power radiated per electron due to synchrotron radiation is given by

$$
P_g \approx \frac{cC_g}{2p}\frac{E^4}{r^2}; \quad C_g \approx 8.85\times10^{-5}\,\mathrm{GeV^{-3}m}
$$

Hence

$$
P_g \approx \frac{cC_g}{2p}\times\frac{10^4}{(6.4\times10^6)^2} \approx 1\ \mathrm{keV}
$$

***Q2**: For a fixed beam power and centre of mass energy, show that the luminosity scaling for a round beam ($s_x = s_y = s$) at the IP can be expressed as*

$$
L \propto \frac{s_z}{N_b} d_{SB}
$$

Comment on this result.

We begin with the relationship

$$L \propto \frac{N_b}{s_x s_y} = \frac{N_b}{s^2}$$

The energy loss due to beamstrahlung scales as

$$d_{SB} \propto \frac{N_b^2}{s_z(s_x + s_y)^2} \propto \frac{N_b^2}{s_z s^2}$$

Combining the two gives

$$L \propto \frac{s_z}{N_b} d_{SB}$$

which is independent of the beam size. This would seem to suggest that lowering bunch current will increase the luminosity, but we should remember that we are holding the power constant, and so we would need to also increase either the number of bunches in the train or the repetition rate. Reducing the bunch charge allows you to decrease the beam size for a fixed beamstrahlung.

Q4: Keeping all other parameters in our final luminosity scaling law constant, how does the beamstrahlung scale with centre of mass energy?

For a flat beam, we have:

$$d_{SB} \propto \frac{E_{cm} N_b^2}{s_z s_x^2}$$

Remembering that

$$s_x^2 = e_{x,n} b_x / g \propto \frac{1}{E_{cm}}$$

we immediately see that $d_{BS} \propto E_{cm}^2$.

Q6: The shunt impedance per unit length of an NLC structure (f = 11 GHz) is ~80 MW/m. The attenuation factor is ~0.5 and the structure length is 0.9 m. Calculate the peak RF power required to produce a gradient of 65 MV/m. Assume that the gradient is constant along the length of the structure. Assuming Q ~ 8300, estimate the filling time of the structure. What can you infer about the average group velocity of the structure?

We start from

$$\frac{dP}{dz} = -\frac{E_0^2}{r_s}$$

In addition we know that

$$P_L = P_0 e^{-2t_0}$$

Assuming E_z and therefore *dP/dz* to be constant along the structure (constant gradient structure), we can write

$$\begin{aligned}\frac{dP}{dz} &= \frac{P_L - P_0}{L} \\ &= \frac{P_0}{L}\left(e^{-2t_0} - 1\right)\end{aligned}$$

Combining and rearranging, we arrive at an expression for the required input power:

$$P_0 = \frac{E_z^2 L}{r_s\left(1 - e^{-2t_0}\right)}$$

Putting in the above approximate numbers for NLC, we arrive at

$$P_0 \approx 75\mathrm{MW}$$

The fill time is given by

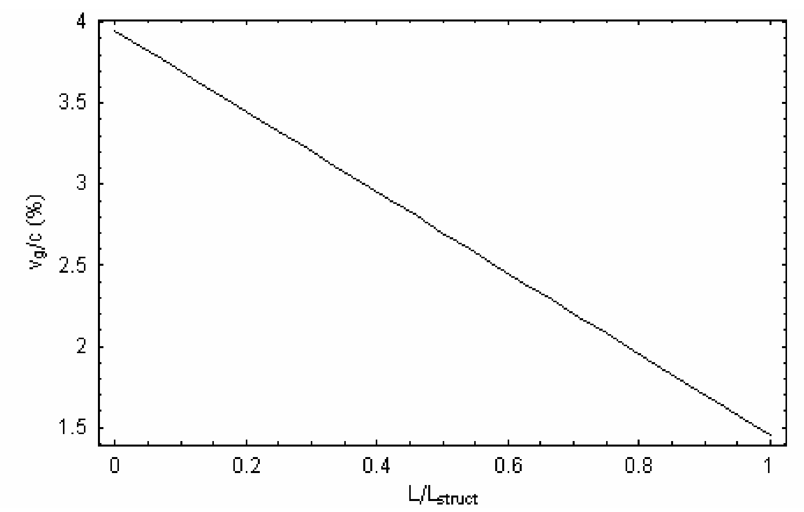
$$t_f = t_0 \frac{2Q}{w} \approx 120 \text{ ns}$$

The average group velocity is $v_g \approx L/t_f \approx 0.025c$.

The actually group velocity changes linearly along the structure:

$$v_g(z) = \frac{w}{Q}\frac{L - (1 - e^{-2t})z}{1 - e^{-2t}}$$

The figure below shows the group velocity along the structure. The average is 2.7% of *c*, which is slightly higher than our previous estimate of 2.5%.



Q7: LEP as a damping ring.

Calculate the damping time for LEP with a beam energy of 90 GeV, and a radius of 4.3 km.

The average power radiated per electron is given by $P_g = \frac{cC_g}{2p}\frac{E^4}{r^2}$

$$P_g \approx \frac{(3\times10^8)\times(8.85\times10^{-5})}{2p}\times\frac{90^4}{4300^2} \approx 15000\,\text{GeV/s}$$

Hence the damping time is given by

$$t_D \approx \frac{2\times90}{15000} \approx 12\text{ms}$$

Q8: The TESLA damping ring is extremely unconventional due to the very long bunch train (285 km), which is 'compressed' by a factor ~17 by injecting and ejecting individual bunches using fast kickers. The 'ring' circumference is approximately 17 km, made up of two arc sections, each of radius 160 m, connected to each other by two long straight sections. The ring energy is 5 GeV.

a) Estimate the damping time of the bare ring (i.e. with no wigglers).
b) A damping time of ~30 ms is required. Assuming wigglers with an RMS field of 2 Tesla, estimate the total length of wiggler needed.

For the purposes of this example, you can assume that the two 'arcs' form almost complete rings.

For a 30 ms damping time, we require an average power radiated of $P_g = 2E/t = 333.3$ GeV/s. The energy lost per turn is therefore 17 km $\times$ 333.3 GeV/s/c $\approx$ 18.9 MeV.

The energy lost per turn in a single arc

$$\Delta E_{turn} \approx C_g\,\frac{E^4}{r} = (8.85\times10^{-5})\times5^4/160 \approx 345.7\text{ kV}.$$

The total energy lost per turn in the wigglers is 18.9 - 2×0.346 ≈ 18.2 MeV. The energy lost in a wiggler is

$$\Delta E_w(\text{GeV}) = 1.27\times10^{-6}B^2(\text{T})E^2(\text{GeV})L(\text{m})$$

Hence the total length of wiggler is $0.0182/(1.27\times10^{-6}\times2^2\times5^2) \approx 143\,\text{m}$.

Q10*: A damping ring at 2 GeV produces a bunch that is 5 mm long and has an relative energy spread of 0.1%. Assuming an L band compressor at 1.3 GHz, estimate the RF voltage required to compress the bunch to 0.1 mm. What is the required* R_{56}*?*

$$V_{RF} \approx \frac{E}{k_{RF}} \left(\frac{d_u}{s_{z,0}} \right) r_c$$

r_c = 5/0.1 = 50. k_{RF} = 2*pf/c* = 27.23 m^{-1}. Thus

$$V_{RF} \approx \frac{2}{27.23} \times \left(\frac{0.001}{0.005} \right) \times 50 \approx 734\,\mathrm{MV}\ .\ R_{56} \approx \frac{E}{k_{RF} V_{RF}} \approx 0.1\,\mathrm{m}.$$

***Q11**: The final energy spread from the compressor in Q9 is 5% which is very large and will cause problems for chromatic emittance growth. One way around this is to use a two stage compressor, with some acceleration between the two to adiabatically damp the energy spread from the first compression before performing the second. Perform the following steps:*

a) *Repeat Q9 for a compression ration of 10.*
b) *assuming the beam is then accelerated to 8 GeV, calculate the bunch compressor parameters for the final compression to 0.1 mm. What is the final energy spread? (For this example, you may assume the 6 GeV acceleration is on crest and is uniformly applied to the entire bunch).*

What is the total longitudinal phase space rotation of the complete system. What implications does this have for damping ring ejection phase errors?

For a compression ratio of 10, we can simply scale the previous results. The required voltage $V_{RF} = 734/5 \approx 147$ MV . From $R_{56} \approx \frac{E}{k_{RF} V_{RF}}$ it immediately follows that the R_{56} must increase by a factor 5: hence $R_{56} \approx 0.5$ m .

The final energy spread from this first stage compression is $10 \times 0.1 = 1\%$, and the final bunch length is 0.5 mm. We now accelerate the bunch from 2 GeV to 8 GeV. The energy spread then damps to 2/8×1% = 0.25%. We use this as input to the next stage compressor.

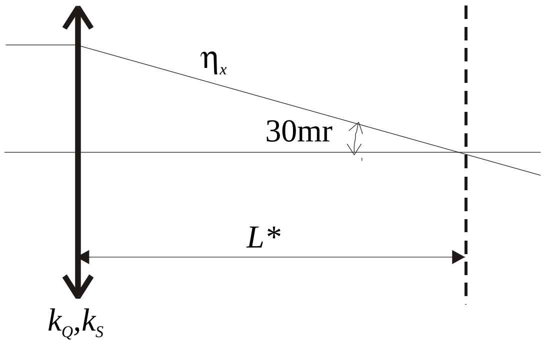
Our input parameters are now:

E = 8 GeV
$s_{z,0} = 0.5\mathrm{mm}$
$d_u = 0.25\%$
$r_c = 5$ (to achieve 0.1 mm)

Following the same procedure as before, we get $V_{RF} \approx 7.3\mathrm{GeV}$, and $R_{56} \approx 0.04$ m . The final energy spread is 1.25% which is a factor of 4 reduce from our single stage solution. Unfortunately, this reduction comes at the expense of considerable more (total) RF voltage.

This two stage system represents a total of 180° rotation of longitudinal phase space. As a consequence, any phase error at the entrance transforms into a phase error at the exit (linac entrance). Phase errors at the entrance of the linac correspond to an energy error, since this error (Δf) does not change along the entire linac. For this reason, the NLC actually uses a phase *telescope* system as the second stage, which has a net rotation of zero (actually it is +90° followed by -90°). The total rotation is therefore 90°. Hence phase errors from the damping ring become energy errors at the linac entrance, but there is no phase error. This initial energy error adiabatically damps away along the linac and is not such a problem as the phase error.

***Q12**: A final focus system has an L^* = 3 m, and a $b^*_{x,y}$ = 15, 0.3 mm. The (geometric) emittances are $2´10^{-11}$ m and $6´10^{-14}$ m in the horizontal and vertical planes. Assuming a simple thin-lens model, where a thin lens quadrupole and a thin lens sextupole are placed exactly at L*, calculate the sextupole strength needed to cancel the chromaticity (assume the horizontal dispersion has an angle of 30 mrad at the IP). Taking the remaining geometric (d independent) vertical kick from the sextupole, estimate the relative increase in vertical IP beam size from this aberration. For an RMS energy spread of 0.3%, estimate relative horizontal beam size increase from the second-order dispersion term.*



The sextupole strength is given by $k_S = k_Q / h$. We know that $K_Q = 1/f = 1/L^*$, and that $h = L^* h'^*$: hence $K_S = \dfrac{1}{L^{*2} h'^*} \approx \dfrac{1}{3^2 \times 0.03} \approx 3.7\,\mathrm{m}^{-2}$.

The geometric vertical kick from the sextupole is

$$\Delta y' = k_S x y$$

where *x*, *y* are the particle coordinates at the sextupole. This kick generates a small additional offset (aberration) at the IP of $\Delta y^* = L^* \Delta y' = L^* k_S x y$. We need to calculate the RMS of this aberration. Assuming the particle coordinates at the sextupole are uncorrelated (i.e. no coupling)

$$\left\langle \Delta y^{*2} \right\rangle = L^{*2} k_S^2 \left\langle x^2 y^2 \right\rangle = L^{*2} k_S^2 \left\langle x^2 \right\rangle \left\langle y^2 \right\rangle$$

Remembering we effectively focus the beam to a point, we can write $\langle x^2 \rangle = q_x^{*2} L^{*2}$ and $\langle y^2 \rangle = q_y^{*2} L^{*2}$, where $q_{x,y}^*$ are the betatron beam divergences at the IP given by $q_{x,y}^* = \sqrt{e_{x,y} / b_{x,y}^*}$. Putting this all together and dividing the left and right hand side by $s_y^{*2} = e_y b_y^*$ we arrive at

$$\left(\frac{\Delta y_{\mathrm{RMS}}^*}{s_{\mathrm{y}}^*} \right)^2 = \frac{L^{*6} k_S^2 e_x}{b_x^* b_y^{*2}}$$

$$\frac{\Delta y_{\mathrm{RMS}}^*}{s_{\mathrm{y}}^*} = \frac{L^{*3} k_S}{b_y^*} \sqrt{\frac{e_x}{b_x^*}}$$

Putting in the numbers we find that

$$\frac{\Delta y_{\mathrm{RMS}}^*}{s_y^*} \approx 12$$

Thus, if left uncorrected, the geometric sextupole term would increase the vertical beam size by a factor of 12.

The second-order dispersive kick in the horizontal plane generated by the quad/sextupole combination is

$$\Delta x' = -\frac{1}{2} k_Q h d^2 = -\frac{1}{2} h'^* d^2$$

Following the same procedure as before, we write the aberration generated at the IP:

$$\Delta x^* = -\frac{1}{2} L^* h'^* d^2$$

Taking the moments we have

$$\langle \Delta x^{*2} \rangle = \frac{1}{4} L^{*2} h'^{*2} \langle d^4 \rangle$$

Assuming that the energy of the particles has a Gaussian distribution, we can write the fourth-order moment as $\langle d^4 \rangle = 3 d_{\mathrm{RMS}}^4$:

$$\langle \Delta x^{*2} \rangle = \frac{3}{4} L^{*2} h'^{*2} d_{\mathrm{RMS}}^4$$

What we are really interested in is the variance about the mean. Because of the d^2 in the original kick expression, there is a mean value which we must subtract off:

$$\left\langle \Delta x^{*} \right\rangle = \frac{1}{2} L^{*} h'^{*} d_{\mathrm{RMS}}^{2}$$

The variance of the aberration about the mean is given by

$$\begin{aligned} \left\langle \Delta x^{*2} \right\rangle - \left\langle \Delta x^{*} \right\rangle^{2} &= \frac{3}{4} L^{*2} h'^{*2} d_{\mathrm{RMS}}^{4} - \frac{1}{4} L^{*2} h'^{*2} d_{\mathrm{RMS}}^{4} \\ &= \frac{1}{2} L^{*2} h'^{*2} d_{\mathrm{RMS}}^{4} \end{aligned}$$

Finally we estimate the relative size of the RMS aberration as before:

$$\frac{\Delta x_{\mathrm{RMS}}^{*}}{s_{\mathrm{x}}^{*}} \approx \frac{L^{*} h'^{*} d_{\mathrm{RMS}}^{2}}{\sqrt{2}\sqrt{e_{x} b_{x}^{*}}}$$

For the values quoted, this gives us

$$\frac{\Delta x_{\mathrm{RMS}}^{*}}{s_{x}^{*}} \approx 1$$

So the size of the aberration is roughly equal to the nominal beam size. Since the aberration adds in quadrature to the nominal beam size, this corresponds to a ~40% increase.

***Q13**: For a normalised emittance of 30 nm, calculate the Oide limit.*

$$\begin{aligned} s_{y,\mathrm{min}}^{*} &\approx 1.83 \left(r_{e} \lambda\!\!\!^{-}_{e} F \right)^{1/7} e_{y,n}^{5/7} \\ b_{y,\mathrm{min}}^{*} &\approx 2.39 \left(r_{e} \lambda\!\!\!^{-}_{e} F \right)^{2/7} e_{y,n}^{3/7} g \end{aligned}$$

Taking the following numbers:

F = 7 (typical number)
$\lambda\!\!\!^{-} = 3.86 \times 10^{-13}$ m
$r_{e} = 2.82 \times 10^{-15}$ m
$g = 5 \times 10^{5}$ (~ 250 GeV beam energy)

we arrive at

$$\begin{aligned} s_{y,\mathrm{min}}^{*} &\approx 1.4\,\mathrm{nm} \\ b_{y,\mathrm{min}}^{*} &\approx 0.024\ \mathrm{mm} \end{aligned}$$