An Introduction to the
Physics and Technology
of e+e- Linear Colliders

Lecture 2: Main Linac

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Maxwell's Equations in MKSA units

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Where:

$$\vec{B} = \mu \vec{H}$$

$$\vec{D} = \varepsilon \vec{E}$$

Wave Equations for E and H

Make use of a vector calculus identity:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) \equiv \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A}$$

Apply to the Electric field curl equation, and assume no charges or currents are present:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \right) \equiv \vec{\nabla} \left(\vec{\nabla} \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

zero (no charges present)

Reverse Order of Derivatives

$$\nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{B} \right)$$

Wave Equations for E and H (2)

Replace B with μH, and replce curl of B via Maxwell's magnetic curl equation:

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{H} \right)$$

$$= \mu \left(\frac{\partial^2 \vec{D}}{\partial t^2} + \frac{\partial \vec{J}}{\partial t} \right)$$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

And similarly for the magnetic field:

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Free-Space Solution to the Wave Eqn

Let us define the z-axis parallel to the direction of propagation, and solve the wave equation as a superposition of travelling plane waves:

$$\vec{E} = \vec{E_0} \exp\left[i\left(\omega t - kz\right)\right], \vec{H} = \vec{H_0} \exp\left[i\left(\omega t - kz\right)\right]$$

Absence of boundaries (isotropic, homogeneous vacuum) requires that vectors E_0 and H_0 be constant for all time and space

Applying the wave equation to the electric field yields:

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2}{\partial t^2}\left\{\vec{E_0} \exp\left[i\left(\omega t - kz\right)\right]\right\}$$

Free-Space Solution to the Wave Eqn (2)

Since E_0 is constant, both the time derivative and Laplacian operate only on the complex exponential. After cancellation of constant factors, we find:

$$k^2 = \mu\varepsilon\omega^2 \rightarrow \frac{\omega}{k} = \frac{\partial\omega}{\partial k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

i.e., a plane wave with a phase velocity and a group velocity = $1/\sqrt{\mu\varepsilon}$. In vacuum, then, a wave which propagates at v=c.

Free-Space Solution to the Wave Eqn (3)

Go back and apply Maxwell's equations to this solution:

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial \vec{E}}{\partial z} = -ikE_{0,z} \exp\left[i\left(\omega t - kz \right) \right] = 0$$

One of these (k or $E_{0,z}$) must be zero!

k=0: trivial solution, no wave!

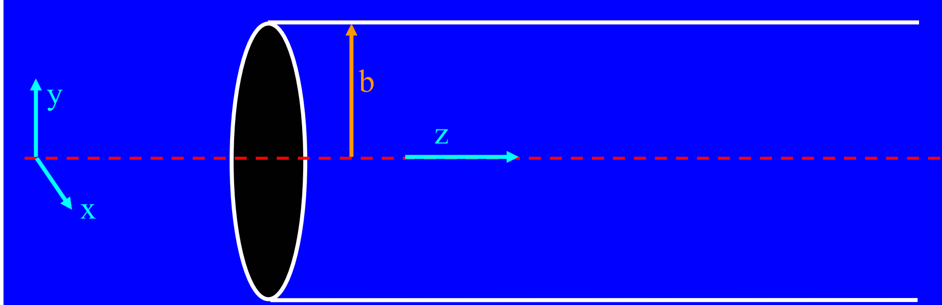
$E_{0,z} = 0$: electric field accelerates beam transverse to direction of wave propagation! If beam is accelerated in x while wave moves in z, then wave will first accelerate, then decelerate, the beam!

No good for acceleration!

Bounded Solution to the Wave Equation

Next simplest solution: apply some kind of boundaries in x and y, so that non-zero x and y derivatives of the electric field can cancel z derivative (ie, permits non-zero $E_{0,z}$ while still obeying Maxwell).

Try a conducting pipe of radius b, oriented along z axis:



Circular Waveguide Solution

Once again,

$$\vec{E} = \vec{E}_0 \exp\left[i\left(\omega t - kz\right)\right], \vec{H} = \vec{H}_0 \exp\left[i\left(\omega t - kz\right)\right]$$

This time vectors E_0 and H_0 are functions of transverse coordinates x and y (or r and θ) but not z or t. Thus we can simplify some derivatives:

$$\frac{\partial}{\partial z} = -ik, \frac{\partial^2}{\partial z^2} = -k^2$$
$$\frac{\partial}{\partial t} = +i\omega, \frac{\partial^2}{\partial t^2} = -\omega^2$$

Circular Waveguide Boundary Conditions

At the boundary (r=b), the *normal* component of B and the *tangential* component of E are continuous. If the conductor is perfect, then *within* the conductor the electric and magnetic field are identically zero. Thus at r=b, H_r, E_z, and E_θ → 0.

Since E_θ=0, the θ component of the magnetic curl equation must go to zero. In total, then:

$$\left.\begin{aligned} E_\theta &= 0 \\ E_z &= 0 \\ H_r &= 0 \\ \frac{\partial H_z}{\partial r} &= 0 \end{aligned}\right. \quad @\ \mathrm{r} = \mathrm{b}$$

Circular Waveguide: Longitudinal Fields

Rewrite the z component of the wave equation:

$$\nabla_{\perp}^{2} E_z - k^2 E_z + \mu\varepsilon\omega^2 E_z = 0$$

Now we can:

.Cancel common complex exponent factor from E_z

.Define $k_c^2 \equiv \mu\varepsilon\omega^2 - k^2$

$$\nabla_{\perp}^{2} E_{0,z} + k_c^2 E_{0,z} = 0$$

$$E_{0,z} = \sum_{n=0}^{\infty} a_n J_n(k_c r)\cos(n\theta + \theta_n)$$

Applying Boundary Conditions

1. n is an integer because field is single-valued -- $\cos[n(\theta+2\pi)] = \cos n\theta$ iff n is integer.

2. To force $E_z \rightarrow 0$ @ r=b, $k_c b = z_{np}$, where z_{np} is the p[th] zero of J_n. Note that $k_c > 0$ as a result, and

$$E_{0,z} = \sum_{p=1}^{\infty}\sum_{n=0}^{\infty} a_{np} J_n(k_{c,np} r)\cos(n\theta + \theta_{np})$$

Note that: $k_{c,np} = \dfrac{z_{np}}{b} = \sqrt{\mu\varepsilon\omega^2 - k^2}$ So k=0 must correspond to a nonzero ω:

$$\omega_{c,np} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{z_{np}}{b}$$ Cutoff Frequency

Physical Meaning of Cutoff Frequency

$\omega > \omega_c$: Real-valued k is possible, wave is complex exponential

$\omega < \omega_c$: k is imaginary, wave falls off exponentially with z (can't propagate) -- *evanescent* wave!

Calculate phase and group velocity of the wave:

$$v_{gr} = \frac{\partial \omega}{\partial k} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{\sqrt{\omega^2 - \omega_c^2}}{\omega} < c$$

$$v_{ph} = \frac{\omega}{k} = \sqrt{\frac{1}{\mu\varepsilon} + \frac{\omega_c^2}{k^2}} > c$$

Since particle moves < c, wave will overtake particle -- no acceleration possible!

TE and TM modes

The longitudinal electric field vector is given by:

$$E_{0,z} = \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} a_{np} J_n\left(k_{c,np} r\right) \cos\left(n\theta + \theta_{np}\right)$$

A similar solution is available for the magnetic field vector:

$$H_{0,z} = \sum_{v=1}^{\infty} \sum_{u=0}^{\infty} f_{uv} J_u\left(k_{c,uv} r\right) \cos\left(u\theta + \theta_{uv}\right)$$

In this case the BC's require that dH/dr→0, so different cutoff frequency from electric field.

TE and TM modes (2)

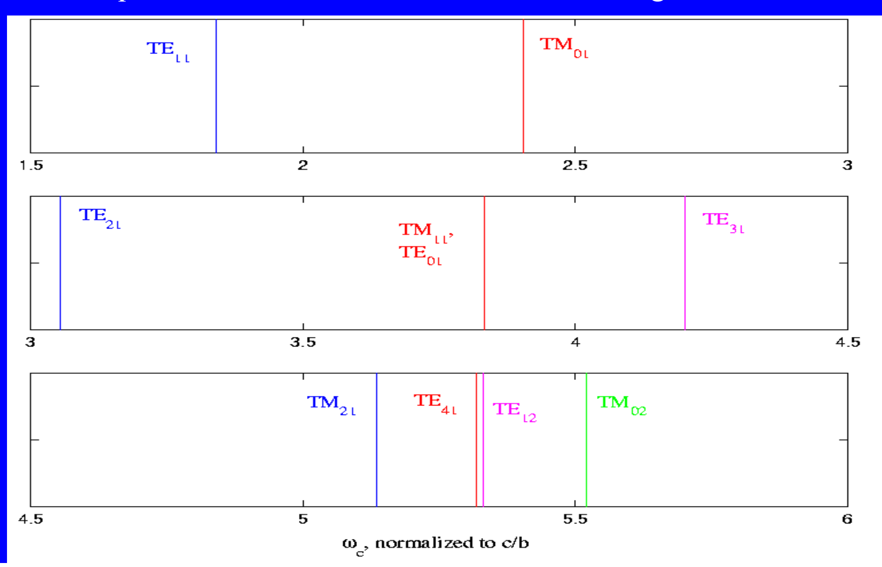
Since J and J' have different zeroes, in general waves with nonzero $H_{0,z}$ have different phase/group velocities and cutoff frequencies from waves with nonzero $E_{0,z}$. Thus, *in general a wave with a given phase and group velocity cannot have both a longitudinal electric field and a longitudinal magnetic field!*

Waves with $H_{0,z} \equiv 0$ are called TM (transverse magnetic) modes; waves with $E_{0,z} \equiv 0$ are called TE (transverse electric) modes. Usually the modes are referred to with their index numbers, TE_{uv} or TM_{np}.

Note: TM_{01} mode has nonzero E_z, E_r, H_θ components only!

TE and TM Modes (3)

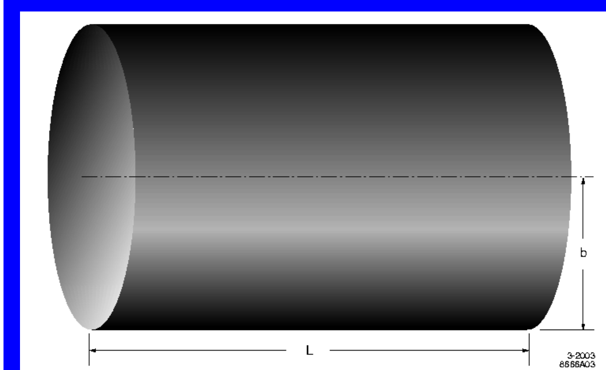
Cutoff Frequencies for a few modes in circular waveguide:



So where are we?

- Considered waves in free space and in regular cylindrical waveguide
- Waves in free space are no good for acceleration
  - electric field ⊥ momentum vector
- Waves in cylindrical waveguide also no good
  - can get electric field parallel to momentum vector, but…
  - electric field phase velocity > c, wave overtakes particle
- Can force modes in waveguide to work by separating beam from wave after < 1 wavelength
  - requires addition of boundary conditions in z!!!!
  - How about perfectly-conducting end walls?

Waves in a Conducting Hollow Cylinder



In addition to existing BC's, we find:

.At z=0 and z=L, E_r and $E_\theta \rightarrow 0$ (E normal to any conducting boundary)

. At z=0 and z=L, $H_z \rightarrow 0$ (H tangential to any conducting boundary)

.$dH_\theta/dz = dH_r/dz = 0$ at ends

TM_{010} (Standing Wave) Solution

If we consider a rightward (+z)-propagating TM_{01} wave and a leftward-propagating wave with the same frequency and amplitude, the nonzero field vectors become:

$$E_z = J_0\left(k_{c,01} r\right)\cos(kz)\exp(i\omega t)$$

$$E_r = -\frac{k}{k_{c,01}} J'_0\left(k_{c,01} r\right)\sin(kz)\exp(i\omega t)$$

$$H_\theta = \frac{-i\omega\varepsilon}{k_{c,01}} J'_0\left(k_{c,01} r\right)\cos(kz)\exp(i\omega t)$$

TM_{010} (2)

The z=0 and z=L boundary conditions are automatically satisfied iff:

$$kL = j\pi,\ j \in \mathbb{Z}$$

This wave is usually called a TM_{npj} mode, and only certain discrete frequencies

$$\omega_{npj} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{z_{np}}{b}\right)^2 + \left(\frac{j\pi}{L}\right)^2}$$

are permitted in this mode.

Properties of TM_{npj} Modes

1. Since $J_n(0) \neq 0$ iff n=0, n=0 modes are optimal for acceleration (otherwise you have to put your particle far from the center of the cavity!)

2. The rightward-travelling component of the standing wave will always accelerate the beam (assuming an appropriate choice of phase w.r.t. the beam arrival time).

3. For most combinations of parameters, the leftward-travelling component of the standing wave will decelerate the beam if j>0, hence j=0 is optimal for real-world applications.

4. For a TM_{0p0} mode cavity, if the field is given by $E_z = E_0 \exp(i\omega t)$ the maximum energy gain achievable by a particle is E_0LT, where

$$T \equiv \frac{\sin(\psi/2)}{\psi/2}$$

and ψ = the phase variation of the wave during the particle's passage (*transit angle*)

Stored Energy in a Cavity; R/Q

The stored energy in an AC electromagnetic field is given by:

$$\vec{E}\cdot\vec{D}^* = \varepsilon\left[E_0 J_0\left(k_{c,0p}r\right)\right]^2 \cos^2 \omega t$$

$$U = \frac{1}{2}\left(\vec{E}\cdot\vec{D}^* + \vec{B}\cdot\vec{H}^*\right)$$

$$\vec{B}\cdot\vec{H}^* = \mu\left[\frac{\omega\varepsilon E_0}{k_{c,0p}} J'_0\left(k_{c,0p}r\right)\right]^2 \sin^2 \omega t$$

To simplify the problem, choose t=0 so all energy is in electric field!

Stored Energy and R/Q (2)

When we integrate the electric field energy over the volume, we find a relationship between the max acceleration $V \equiv E_0 LT$ and the stored energy:

$$\frac{V^2}{U} = \frac{2T^2}{\pi \varepsilon J_1^2(z_{0p})} \frac{L}{b^2}$$

If we introduce the *impedance of the medium* $Z \equiv \sqrt{(\mu/\varepsilon)}$ (vacuum: 377Ω) and perform a few simplifications and substitutions, then:

$$\frac{V^2}{\omega_{0p0} U} = \frac{T^2 L}{b} Z \equiv \frac{R}{Q}$$

("R over Q" or "R upon Q") -- a purely geometric quantity which relates stored energy, accelerating voltage, and frequency

Stored Energy and R/Q (3)

With appropriate substitutions, we can rewrite R/Q:

$$\frac{R}{Q} = Z \frac{2}{z_{0p}} \frac{\sin^2(\psi/2)}{\psi/2} c\sqrt{\mu\varepsilon}$$

R/Q is maximized when $\psi \sim 134°$. This is less useful than it sounds, but in general the stored energy can be minimized by optimizing the transit angle and increasing the frequency (recall that $R/Q = V^2/\omega U$).

Furthermore, since z_{0p} increases monotonically with p, selection of p=1 will also minimize the stored energy required for a given voltage, and we will assume such a selection henceforth.

Extending to Finite Conductivity

What happens when there is some small resistance to the surface of the cavity?

In this case the electric and magnetic fields in the conductor are nonzero.

Since the tangential magnetic field is continuous at the boundary, the tangential field no longer vanishes thereat.

After a great deal of mathematics, the ohmic losses (product of the current density and the electric field in the conductor) can be expressed in terms of the tangential magnetic field at the surface of the cavity:

$$\frac{dP}{dA} = \sqrt{\frac{\mu_c \omega}{2\sigma}} \left| \vec{H}_S \right|^2$$

$\mu_c \equiv$ permeability of conductor ($\cong \mu_0$)

$\sigma \equiv$ conductivity of conductor ($1/\Omega$m)

Finite Conductivity (2)

We can usefully rewrite the power dissipation equation:

$$\frac{dP}{dA} = R_S(\omega) \left| \vec{H}_S \right|^2$$

where R_S = frequency-dependent conductivity ("surface resistance")

$$R_S \equiv \sqrt{\frac{\mu_c \omega}{2\sigma}}$$

Normal conductors

$$R_S[\Omega] \cong 9 \times 10^{-5} \frac{\upsilon^2 [GHz]}{T[K]} \exp\left(-\alpha_{SC} \frac{T_C}{T} \right) + R_{RES}$$

Superconductors

$\alpha_{SC} \sim 1.92$, $T_C = 9.2$ K (Niobium), $R_{RES} \sim 10^{-8}\ \Omega$

Wall Q of a Cavity

If we return to our right-circular cylinder example and integrate the ohmic losses over the endcaps and the barrel, we arrive (after some Bessel function tricks):

$$P_{cav} = \frac{\pi E_0^2 R_S}{Z^2} J_1^2(z_{01}) b(L+b)$$

P, U both ~ E² → P ~ U

Define:

$$Q_W \equiv \frac{\omega U}{P} = \frac{z_{01} Z L}{2 R_S (L+b)}$$

"Wall Q," fractional loss of stored energy into walls in 2π x RF period.

Shunt Impedance

Now that we have power loss per unit energy, and voltage per unit energy, we can compute relationship between available input power and achievable voltage (voltage at which input power = wall losses). This relationship has ohmic units, hence is called *Shunt Impedance:*

$$R_{cav} = Q_w \frac{R}{Q}$$

$$= \frac{V^2}{P}$$

$$= \frac{Z^2}{R_S} \frac{2c^2 \mu\varepsilon}{z_{01}} \frac{\sin^2(\psi/2)}{1 + c\sqrt{\mu\varepsilon} \frac{2}{z_{01}} \frac{\psi}{2}}$$

Treat the cavity like a resistor in phys lab! P = V²/R (careful of factor of 2 -- AC power)!

Motivation for multi-celled RF Cavities

Consider a single-celled RF cavity with shunt impedance R. A voltage V can be maintained if power $P = V^2/R$ is supplied to the cavity.

Alternately, if n cavities are available, each one can supply a voltage of V/n at a cost in power of $n(V/n)^2/R = 1/n$ of the power needed to get there in a single cavity.

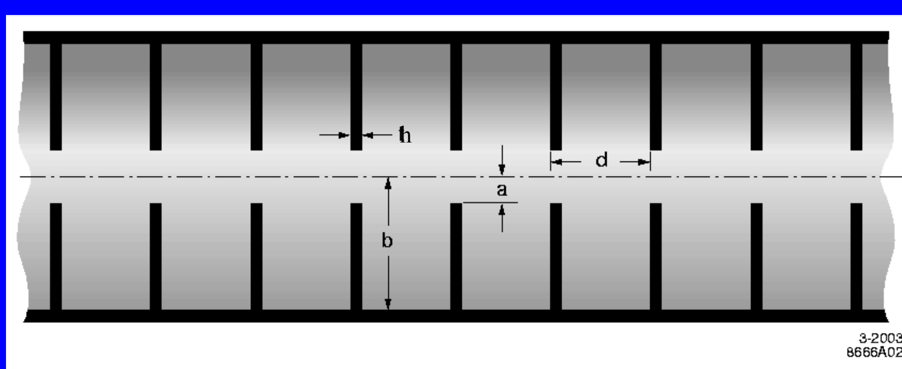
Disadvantage: less stored energy in each cavity means that the beam current that we can accelerate is also reduced. Also, we need more space for the accelerating elements.

However, for 100 GeV and a single 1 GHz cavity, the input power is around 1.5 x 10^{15} watts. California's peak consumption is around 5 x 10^{10} watts.

Apparently multi-celled cavities are a necessity if a voltage of 100 GV is to be achieved in a single pass.

RF Accelerating Structures

Consider an infinite cylindrical waveguide (radius b) divided into RF cavities by plates with thickness h and z-spacing d. In order to allow the beam to pass thru, each plate (or disc) has a hole (iris) of radius a.



RF Accelerating Structures (2)

What do we know about the electric field in an infinite periodic structure? Floquet's theorem tells us:

$$\vec{E}(r,\theta,z+d,t)=\vec{E}(r,\theta,z,t)\exp\left[d\left(-\alpha+ik_z\right)\right]$$

Also, as $a \to 0$, expect the solution to go asymptotically to single-cavity form.

Let us define: $\vec{E}_1, \vec{H}_1$ $\equiv$ Fields in absence of endcap holes

$\vec{E}_2, \vec{H}_2$ $\equiv$ Fields with endcap holes included

And consider the expression: $\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)$ (Yuk)

RF Accelerating Structures (3)

$$\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)=\int dVol\,\vec{\nabla}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)$$

$$=\int dVol\left[\vec{H}_2^{*}\cdot\left(\vec{\nabla}\times\vec{E}_1\right)-\vec{E}_1\cdot\left(\vec{\nabla}\times\vec{H}_2^{*}\right)+\vec{E}_2\cdot\left(\vec{\nabla}\times\vec{H}_1^{*}\right)-\vec{H}_1^{*}\cdot\left(\vec{\nabla}\times\vec{E}_2\right)\right]$$

$-\mu\dfrac{\partial\vec{H}_1}{\partial t}=-i\mu\omega_1\vec{H}_1$, etc.

After a bunch of such substitutions, collection of terms, and asserting that perturbed and unperturbed fields are ~ equal:

$$\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)=i\left(\omega_2-\omega_1\right)\int dVol\left(\mu\vec{H}_1^{*}\cdot\vec{H}_1+\varepsilon\vec{E}_1^{*}\cdot\vec{E}_1\right),\text{ or}$$

$$\left(\omega_2-\omega_1\right)=\frac{-i\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^{*}-\vec{E}_2\times\vec{H}_1^{*}\right)}{2U_1}$$

RF Accelerating Structures (4)

$$(\omega_2 - \omega_1) = \frac{-i\oint d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^* - \vec{E}_2\times\vec{H}_1^*\right)}{2U_1}$$

$$d\vec{A}\cdot\left(\vec{E}_1\times\vec{H}_2^*\right) = 0$$

Electric field is normal to the surface so parallel do the differential area vector

$\vec{H}_1$ Is known at all points from the TM_{010} solution

$\vec{E}_2 \approx \vec{E}_1$ Except in the vicinity of the hole

In principle, it's possible to calculate the frequency of the system (w.r.t. the unperturbed TM_{010} frequency) after a lot of arithmetic and calculus

Accelerating Structure Dispersion Relation

$$\omega = \frac{z_{01}}{b\sqrt{\mu\varepsilon}}\left\{1 + \frac{2}{3\pi J_1^2(z_{01})}\frac{a^3}{b^2 d}\left[1 - \exp(-z_{01}h/a)\cos(k_z d)\right]\right\}$$

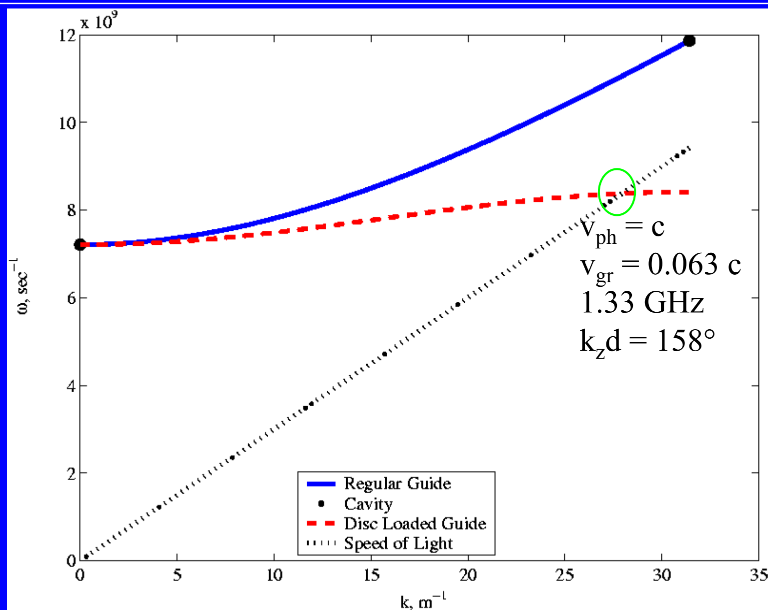
$$v_{ph} = \frac{z_{01}}{k_z b\sqrt{\mu\varepsilon}}\left\{1 + \frac{2}{3\pi J_1^2(z_{01})}\frac{a^3}{b^2 d}\left[1 - \exp(-z_{01}h/a)\cos(k_z d)\right]\right\}$$

$$v_{gr} = \frac{z_{01}}{b\sqrt{\mu\varepsilon}}\frac{2}{3\pi J_1^2(z_{01})}\frac{a^3}{b^2}\exp(-z_{01}h/a)\sin(k_z d)$$

What does it mean? Consider a system with b=10 cm:

.Regular waveguide, or
.Cavity with L = 10 cm, or
.Accelerator Structure with d = 10 cm, a = 5 cm

Disc Loaded Wave Guides

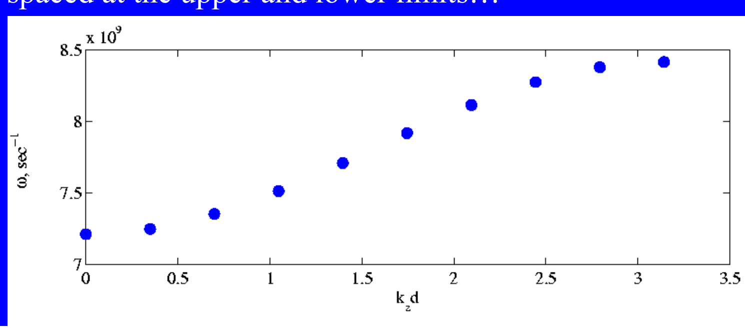


Finite-Length Disc-Loaded Waveguides

Previous estimates of the dispersion relation relied upon the assumption that the structure is essentially infinite.

In real life, a structure with N_{cell} cells acts like an N_{cell}-coupled oscillator system, with N_{cell} oscillation modes.

The modes are equally spaced in k_z, as a consequence the frequencies are closely-spaced at the upper and lower limits…



v_{gr}, v_{ph} and $k_z d$

What is the physical meaning of $k_z d$ in the preceding equations? If we return to Floquet's theorem and assume a simple sinusoidal variation in time, then $E(r,\theta,z+d)\exp(i\omega t) = E(r,\theta,z)\exp(ik_z d)\exp(i\omega t)$

$$E(r,\theta,z+d)\exp[i(\omega t - k_z d)] = E(r,\theta,z)\exp(i\omega t)$$

So a particle with velocity ω/k_z encounters the same phase of RF in each cell ➔ the phase velocity ω/k_z is the *velocity of the synchronous phase.*

At a given time t, then, $k_z d$ is the phase difference between neighboring cells.

Combine those 2 facts, and find *$k_z d$ is synonymous with the transit angle ψ* (though usually called the "phase advance per cell" in this context).

Finally, we assert that v_{gr} is the velocity of energy propagation thru the structure.

$k_z d$ values greater than π

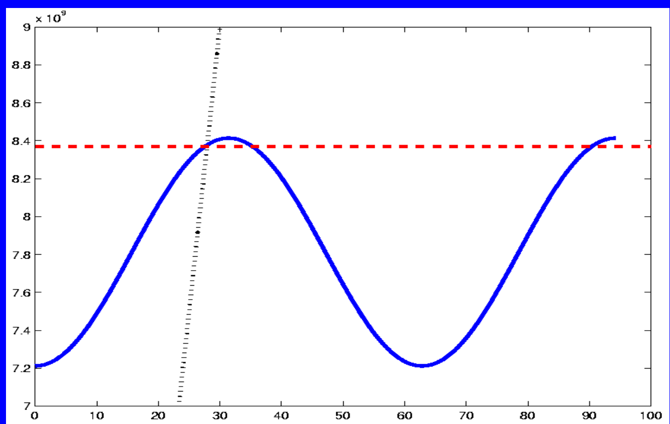
So far, nothing we have talked about precludes a value of $k_z d > \pi$.

Since the dispersion relation is sinusoidal, if we allow $k_z d > \pi$ then the relation becomes multi-valued (many values of k_z correspond to each value of ω)…

All k_z's yield same v_{gr} (with varying signs)

Only one has v_{ph} = c.

What's up with this?



Space Harmonics

Remember that E(z) is a stepwise function (ie, E(z) is constant over 1 cell, then steps to a new value in the next one…)

➔ We can Fourier-decompose it into components with different wavelengths -- but they all oscillate with the same frequency…

➔ The various Fourier components have different phase velocities!

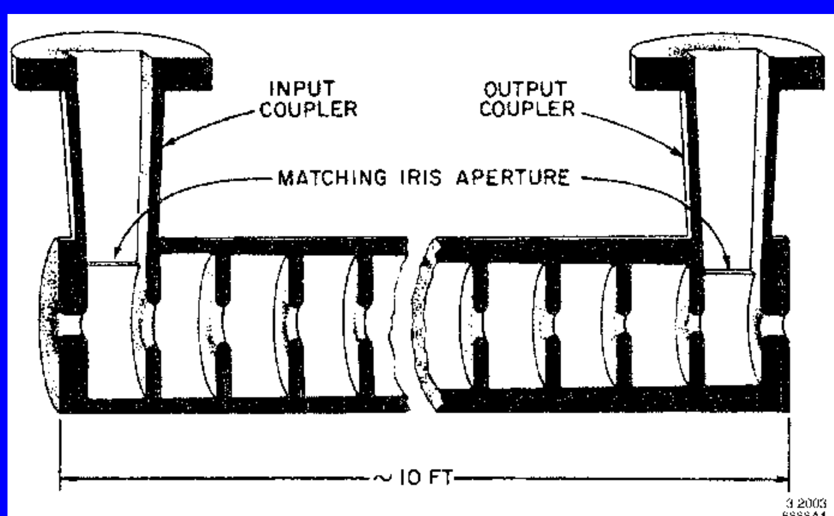
Standard interpretation:

.For a given ω, the corresponding values of $k_d \Leftrightarrow$ the various Fourier components of the accelerating field (*space harmonics*)

.Only the space harmonic at the beam velocity provides net acceleration

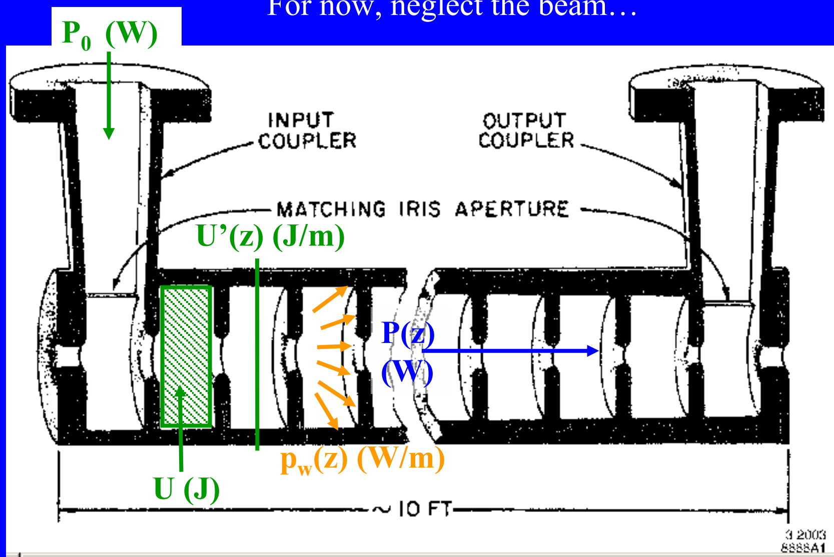
.The effective shunt impedance is reduced because some fraction of RF power (~20-35%) supports non-synchronous harmonics

Travelling-Wave Accelerators



Power Flow through a TW DLWG

For now, neglect the beam…



Power Flow (2)

Conserve Energy: Net power flow must balance net change in stored energy, or:

$$\frac{dU'(z)}{dt} + p_w(z) + \frac{dP(z)}{dz} = 0$$

In steady state: $\frac{dU'(z)}{dt} = 0$

Definition of Q: $p_w(z) = \frac{\omega U'(z)}{Q_w}$

Definition of group velocity: $P(z) = U'(z) v_{gr}(z)$

$$\frac{dP(z)}{dz} = -p_w(z) = -\frac{\omega U'(z)}{Q_w} = \frac{\omega P(z)}{v_{gr}(z) Q_w}$$

Constant Impedance Structure

If v_{gr} is a constant along the structure, then: $P(z) = P_0 \exp(-2\alpha_0 z)$

where $\alpha_0 \equiv \dfrac{\omega}{2Q_w v_{gr}}$

Introduce the gradient $G_0 \equiv dV/dz$

and the normalized shunt impedance $r_l = dR_{struc}/dz$

to find

$$G_0^2(z) = 2\alpha_0 r_l P_0 \exp(-2\alpha_0 z)$$

Constant Impedance Structure (2)

Define $\tau \equiv \alpha_0 L$ (Field e-folding factor per structure)

Total voltage: $V = \sqrt{r_l L P_0}\sqrt{2\tau}\,\dfrac{1-e^{-\tau}}{\tau}$

Filling Time: $t_f = \dfrac{L}{v_{gr}} = \dfrac{2Q_w}{\omega}\tau$

Advantage: All cells are identical (make a zillion copies of 1 design)

Disadvantage: Gradient much higher at front end of structure (bad for gradient-limited structures)

Constant Gradient Structure

Goal: Create a structure such that, when wall losses are taken into account, gradient is (approximately) independent of length

How? Recall: $$\frac{dP(z)}{dz} = -p_w(z) = -\frac{\omega U'(z)}{Q_w} = \frac{\omega P(z)}{v_{gr}(z) Q_w}$$

If Q ~ constant and r_l ~ constant, constant U' ➔ constant gradient

So dP/dz, U',
P(z)/v_{gr}(z) must all be constant!

Constant Gradient Structure (2)

Let us define again τ such that P(z) = $P_0(1-e^{-2\tau})$. Since dP/dz is now a constant,

$$\frac{dP}{dz} = -\frac{P_0}{L}\left(1-e^{-2\tau}\right) = \frac{\omega P(z)}{v_{gr}(z) Q_w}$$

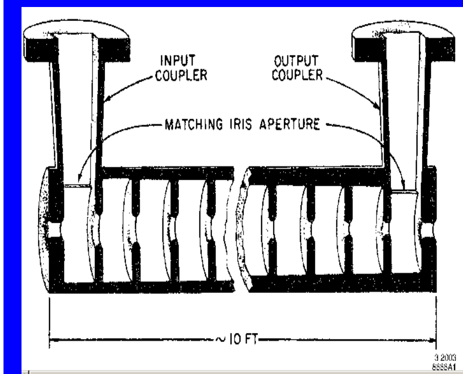
P(z) decreases linearly with z, therefore $v_{gr}(z)$ must do the same:

$$v_{gr}(z) = \frac{\omega}{Q_w} \frac{L - \left(1-e^{-2\tau}\right) z}{1-e^{-2\tau}}$$

$$V = \sqrt{r_l L P_0 \left(1-e^{-2\tau}\right)} \qquad t_f = \frac{L}{v_{gr}} = \frac{2Q_w}{\omega}\tau$$

Standing Wave Accelerator Structures

What happens if we supply input power to the input coupler and the output coupler of a TW structure?



Input coupler power synchronous with beam, accelerates

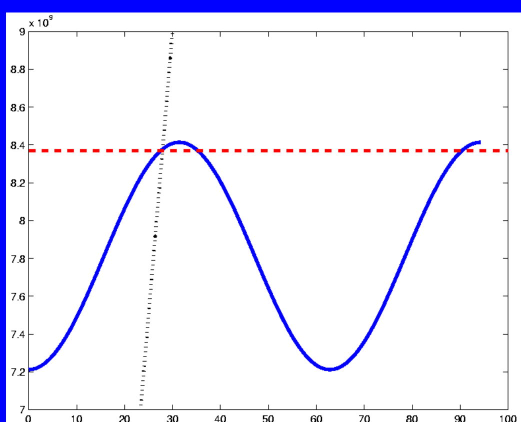
Output coupler power has phase velocity in opposite direction

Backwards power accelerates in some cells, decelerates in others

:: Backwards power is "wasted", shunt impedance reduced by 50%

Standing Wave Accelerator Structures (2)

Exception: Structure that operates in π mode (180°/cell phase advance)



In π mode, "forwards" and "backwards" waves have same phase velocity

Naïve shunt impedance calculation is correct!

Note that group velocity = 0 for π mode -- how's it get energy into the structure?

Filling a Standing-Wave Structure

Recall that structure with N_{cell} cells has N_{cell} normal modes; each mode has its own resonant frequency ω, wave number k_z, all have about the same Q.

When power in the π-mode frequency ω_π hits the input coupler, all the other modes act like resonators driven off-frequency (they are!)

Those other modes transport the power from the input coupler to the cells.

For all practical purposes, it's valid to calculate as though input power at ω_π fills the π-mode instantaneously and uniformly (ie, don't worry about causality!).

Filling a Standing-Wave Structure (2)

Want to trap a lot of energy in the structure, so the TW-style input coupler is no good (lets too much energy out). Use a coupler which lets almost no power through.

Side-effect: lets almost no power in! Takes a long time to fill the structure.

In a filled cavity with no input power, the ratio of power lost out the coupler to power lost into the walls $\equiv \beta_c$ (coupling coefficient)

When the cavity is included, the effective ("loaded") Q is lower:

$$Q_L = \frac{Q_w}{1+\beta_c}$$

Filling a Standing-Wave Structure (3)

Define the characteristic time of a standing wave structure: $t_c \equiv \frac{2Q_L}{\omega}$

Assuming a constant-amplitude of incoming RF (E_{in}), we find:

$$E_e = \frac{2\beta_c}{1+\beta_c} E_{in} \left(1 - e^{-t/t_c}\right)$$

Amplitude of "emitted" wave (RF power leaking back out input coupler)

$$E_{out} = \frac{2\beta_c}{1+\beta_c} E_{in} \left(1 - e^{-t/t_c}\right) - E_{in}$$

Amplitue of emitted + reflected wave (destructive interference)

$$U_c(t) = t_c P_{in} \frac{2\beta_c}{1+\beta_c} \left(1 - e^{-t/t_c}\right)^2$$

Stored Energy

$$V(t) = \left(1 - e^{-t/t_c}\right) \sqrt{\frac{R}{Q} \omega t_c P_{in} \frac{2\beta_c}{1+\beta_c}}$$

Accelerating Voltage

Superconducting Structures

When would you use superconducting materials to make an accelerating structure?

In a TW structure, input power is lost to the walls, the output coupler, and the beam. SC eliminates the first term but the others remain.

For a structure with τ=0.6 (typical), about 70% of the RF power is lost into the walls. Recouping this is a nice idea but maybe not one that justifies the complexity and effort.

Now consider a standing-wave structure: for 1 GHz and $Q_w \sim 10^4$, the e-folding time for stored energy is about 10 microseconds. For $Q_w \sim 10^9$ the e-folding time is about 1 second.

Because there's no output coupler, a SC SW structure is better than a normalconducting one by a factor of 10^5, not 3.