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Abstract

The nonlinear motion of a point charge in the three dimensional space charge field of a Gaussian bunch is analyzed. The 3D space charge field of a Gaussian bunch is derived and the results are compared to the well known Bassetti-Erskine formula for the transverse (2D) space charge fields. The results are applied to the nonlinear motion of an electron in the space charge field of a positron bunch and to the motion of an ion in the space charge field of a electron bunch.

1 Introduction

The beam in a storage ring can interact with the charged particles which can be present in the vacuum chamber. In a positron storage ring an electron cloud can build up in the vacuum chamber due to photoemission or secondary emission while in an electron storage ring ions are created by ionization of the residual gas. In both cases the interaction of the circulating positron or electron beam with a cloud of electrons or ions may result in a degeneration of the beam emittance or in beam instabilities. Simulation codes often use the Bassetti-Erskine formula [1] to calculate in a weak-strong approach the interaction between the bunch and the charged particle [2, 3]. Only the transverse kick is calculated in this approach.

In this report the three dimensional space charge field of a Gaussian bunch is calculated and the results are compared with the Bassetti-Erskine formula. The results are applied to the nonlinear motion of an electron in the space charge field of a positron bunch and to an ion in the space charge field of an electron bunch.

The following situation is considered (see Fig. 1): a Gaussian bunch with total charge Q_b is moving with the velocity v_b along the z-axis of the laboratory frame K. The electric field of the bunch is calculated in the comoving frame $\bar{K}$ and transformed to the laboratory frame K where the Lorentz-Force on a point charge Q_0 is calculated.

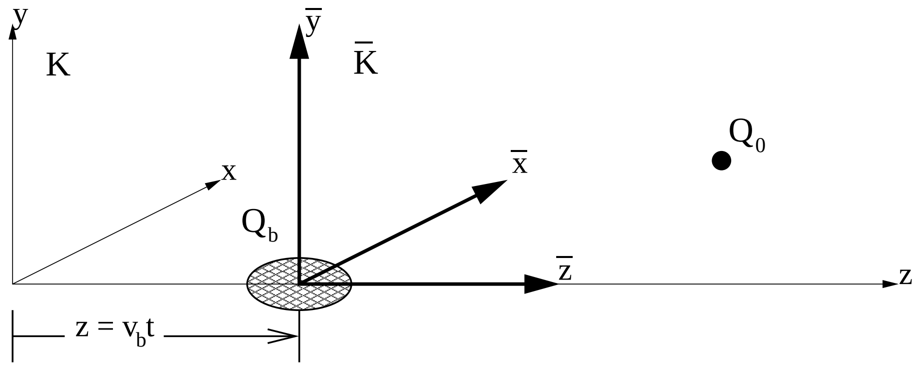

Figure 1: Laboratory frame K and rest frame $\bar{K}$ of a Gaussian bunch

From the velocity of the beam v_b the normalized velocity β_b and the relativistic factor γ_b are calculated as

$$\beta_b = \frac{v_b}{c}, \qquad \gamma_b = \frac{1}{\sqrt{1-\beta_b^2}}. \tag{1}$$

The charge density in the comoving (rest) frame is:

$$\rho(\bar{x},\bar{y},\bar{z}) = \frac{Q_b}{(2\,\pi)^{3/2}\,\sigma_{\bar{x}}\,\sigma_{\bar{y}}\,\sigma_{\bar{z}}}\,\exp\left(-\frac{1}{2}\frac{\bar{x}^2}{\sigma_{\bar{x}}{}^2}\right)\,\exp\left(-\frac{1}{2}\frac{\bar{y}^2}{\sigma_{\bar{y}}{}^2}\right)\,\exp\left(-\frac{1}{2}\frac{\bar{z}^2}{\sigma_{\bar{z}}{}^2}\right), \tag{2}$$

where $\bar{x},\bar{y},\bar{z}$ are the coordinates in the comoving frame $\bar{K}$, which are related to the coordinates in the laboratory frame via a Lorentz-Transformation (see Appendix A) in the following way:

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = \gamma_b\,(z - \beta_b\,c\,t). \tag{3}$$

The beam dimensions in the comoving frame are:

$$\sigma_{\bar{x}} = \sigma_x, \quad \sigma_{\bar{y}} = \sigma_y, \quad \sigma_{\bar{z}} = \gamma_b\,\sigma_z. \tag{4}$$

The bunch is much longer in the comoving frame $\bar{K}$ than in the laboratory frame where the bunch length appears to be Lorentz contracted to the length $\sigma_z = \sigma_{\bar{z}}/\gamma_b$.

The electric space charge fields are calculated from the solution of the Poisson equation in the comoving frame $\bar{K}$:

$$\nabla^2 \Phi(\bar{x}, \bar{y}, \bar{z}) = -\frac{1}{\epsilon_0} \rho(\bar{x}, \bar{y}, \bar{z}). \tag{5}$$

The electric field in the comoving frame $\bar{\boldsymbol{E}} = -\boldsymbol{\nabla}\Phi$ is transformed to an electric and magnetic field in the laboratory frame (see Appendix A):

$$\boldsymbol{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \gamma_b \bar{E}_x \\ \gamma_b \bar{E}_y \\ \bar{E}_z \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} -\gamma_b \beta_b \bar{E}_y / c \\ \gamma_b \beta_b \bar{E}_x / c \\ 0 \end{pmatrix} = \frac{\beta_b}{c} \begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix}. \tag{6}$$

The Lorentz force on a point charge Q_0 with velocity $\boldsymbol{u}$ is:

$$\boldsymbol{F} = Q_0 \left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \right) = Q_0 \begin{pmatrix} (1 - \beta_b u_z / c) E_x \\ (1 - \beta_b u_z / c) E_y \\ E_z + \beta_b \left(E_x u_x / c + E_y u_y / c \right) \end{pmatrix}. \tag{7}$$

The Lorentz force on a comoving point charge, i.e. $u_z/c = \beta_b$, is rather small, since $1 - \beta_b u_z / c = 1/{\gamma_b}^2$, while the force on a colliding point charge ($u_z/c = -\beta_b$) is about a factor two larger than on a point charge at rest. In the next section the fields in the comoving frame $\bar{K}$ are calculated.

2 Electric fields in the rest frame of the bunch

2.1 Solution of the Poisson's equation

Solution of the Poisson's equation (5) for a Gaussian charge distribution (2) is [11] (see Appendix B) :

$$\Phi(\bar{x}, \bar{y}, \bar{z}) = \frac{Q_b}{4\pi\epsilon_0} \frac{1}{\sqrt{\pi}} \int_0^\infty dq \, \frac{\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}} \right)}{\sqrt{q_{\bar{x}} \, q_{\bar{y}} \, q_{\bar{z}}}}, \tag{8}$$

with

$$q_{\bar{x}} = q + 2\,{\sigma_{\bar{x}}}^2, \quad q_{\bar{y}} = q + 2\,{\sigma_{\bar{y}}}^2, \quad q_{\bar{z}} = q + 2\,{\sigma_{\bar{z}}}^2. \tag{9}$$

The potential $\Phi(\bar{x}, \bar{y}, \bar{z})$ is now rewritten in a form which was used in [1] for the two dimensional potential.

The variable q is transformed into the new integration variable ξ according to:

$$\xi^2 = \frac{q_{\bar{y}}}{q_{\bar{x}}} = \frac{q + 2\,{\sigma_{\bar{y}}}^2}{q + 2\,{\sigma_{\bar{x}}}^2}, \quad \text{or} \quad q = \frac{2\,({\sigma_{\bar{x}}}^2 - {\sigma_{\bar{y}}}^2)}{1 - \xi^2} - 2\,{\sigma_{\bar{x}}}^2. \tag{10}$$

Furthermore it is assumed that $\sigma_{\bar{x}} > \sigma_{\bar{y}}$ and the following quantities κ and σ_0 are defined:

$$\kappa = \frac{\sigma_{\bar{y}}}{\sigma_{\bar{x}}}, \quad \sigma_0 = \sqrt{2\,({\sigma_{\bar{x}}}^2 - {\sigma_{\bar{y}}}^2)}. \tag{11}$$

Since $dq/d\xi = 2\, q_{\bar{x}}{}^{3/2} \sqrt{q_{\bar{y}}}\,/\sigma_0^2$ one obtains the following transformed solution of the Poisson's equation:

$$\Phi(\bar{x}, \bar{y}, \bar{z}) = \frac{Q_b}{4\,\pi\,\epsilon_0}\,\frac{2}{\sqrt{\pi}}\,\frac{1}{{\sigma_0}^2}\int_\kappa^1 d\xi\; q_{\bar{x}}\,\frac{\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}}\right)}{\sqrt{q_{\bar{z}}}} \tag{12}$$

with

$$q_{\bar{x}} = {\sigma_0}^2\,\frac{1}{1-\xi^2}, \quad q_{\bar{y}} = {\sigma_0}^2\,\frac{\xi^2}{1-\xi^2}, \quad q_{\bar{z}} = 2\,({\sigma_{\bar{z}}}^2 - {\sigma_{\bar{x}}}^2) + {\sigma_0}^2\,\frac{1}{1-\xi^2}. \tag{13}$$

To make the comparison with the Bassetti-Erskine formula (see Appendix C) easier a constant Φ_0 is defined as

$$\Phi_0 = \frac{Q_b}{2\,\epsilon_0\,\sqrt{\pi}\,\sigma_0}. \tag{14}$$

The definition of Φ_0 may look arbitrary at this point but it will turn out that a simple connection between the two dimensional Bassetti-Erskine formula and the three dimensional results can be established. The potential is now:

$$\Phi(\bar{x}, \bar{y}, \bar{z}) = \Phi_0\,\frac{1}{\sigma_0}\,\frac{1}{\pi}\int_\kappa^1 d\xi\,\frac{q_{\bar{x}}}{\sqrt{q_{\bar{z}}}}\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}}\right) \tag{15}$$

2.2 Electric fields

The electric fields in the comoving frame are now derived from the potential $\Phi(\bar{x}, \bar{y}, \bar{z})$ as:

$$\begin{aligned}
\bar{E}_{\bar{x}} &= -\frac{\partial}{\partial\bar{x}}\Phi(\bar{x}, \bar{y}, \bar{z}) && (16)\\
&= \Phi_0\,\frac{\bar{x}}{\sigma_0}\,\frac{2}{\pi}\int_\kappa^1 d\xi\,\frac{1}{\sqrt{q_{\bar{z}}}}\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}}\right)\\
\bar{E}_{\bar{y}} &= -\frac{\partial}{\partial\bar{y}}\Phi(\bar{x}, \bar{y}, \bar{z}) && (17)\\
&= \Phi_0\,\frac{\bar{y}}{\sigma_0}\,\frac{2}{\pi}\int_\kappa^1 d\xi\,\frac{1}{\xi^2}\,\frac{1}{\sqrt{q_{\bar{z}}}}\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}}\right)\\
\bar{E}_{\bar{z}} &= -\frac{\partial}{\partial\bar{z}}\Phi(\bar{x}, \bar{y}, \bar{z}) && (18)\\
&= \Phi_0\,\frac{\bar{z}}{\sigma_0}\,\frac{2}{\pi}\int_\kappa^1 d\xi\,\frac{q_{\bar{x}}}{q_{\bar{z}}}\,\frac{1}{\sqrt{q_{\bar{z}}}}\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}} - \frac{\bar{y}^2}{q_{\bar{y}}} - \frac{\bar{z}^2}{q_{\bar{z}}}\right)
\end{aligned}$$

with

$$\frac{1}{\sqrt{q_{\bar{z}}}} = \sqrt{\frac{1-\xi^2}{2\,({\sigma_{\bar{z}}}^2 - {\sigma_{\bar{x}}}^2)\,(1-\xi^2) + {\sigma_0}^2}}, \quad \frac{q_{\bar{x}}}{q_{\bar{z}}} = \frac{{\sigma_0}^2}{2\,({\sigma_{\bar{z}}}^2 - {\sigma_{\bar{x}}}^2)\,(1-\xi^2) + {\sigma_0}^2}, \tag{19}$$

and

$$\begin{aligned}
&\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}}-\frac{\bar{y}^2}{q_{\bar{y}}}-\frac{\bar{z}^2}{q_{\bar{z}}}\right) \\
&= \exp\left(-(1-\xi^2)\left(\frac{\bar{x}^2}{{\sigma_0}^2}+\frac{\bar{y}^2}{{\sigma_0}^2}\frac{1}{\xi^2}+\frac{\bar{z}^2}{2\,({\sigma_{\bar{z}}}^2-{\sigma_{\bar{x}}}^2)\,(1-\xi^2)+{\sigma_0}^2}\right)\right). \qquad (20)
\end{aligned}$$

It has been assumed that the transverse beam size does not depend on the longitudinal position. This is a good approximation if one considers only short regions in an accelerator where changes of the beam size due to external magnetic fields can be neglected.

2.3 Approximation of the Transverse Electric fields

For an electron beam energy of (say) 5 GeV the Lorentz factor is large: $\gamma_b \approx 9.8\,10^3$. A bunch length of $\sigma_z = 10\,\text{mm}$ in the laboratory frame corresponds to a bunch length of $\sigma_{\bar{z}} = \gamma_b\,\sigma_z = 98$ m in the comoving frame of the bunch. Therefore the following relation:

$$\sigma_{\bar{z}} \gg \sigma_{\bar{x}} > \sigma_{\bar{y}} \qquad (21)$$

is valid for all ultra relativistic bunches and one obtains

$$\frac{1}{\sqrt{q_{\bar{z}}}} \approx \frac{1}{\sigma_{\bar{z}}\sqrt{2}}, \quad \text{and} \quad \exp\left(-\frac{\bar{z}^2}{q_{\bar{z}}}\right) \approx \exp\left(-\frac{\bar{z}^2}{2\,{\sigma_{\bar{z}}}^2}\right). \qquad (22)$$

In this approximation the transverse electric fields can be written as:

$$\bar{E}_{\bar{x}} = \frac{\Phi_0}{\sigma_{\bar{z}}\sqrt{2\,\pi}}\exp\left(-\frac{\bar{z}^2}{2\,{\sigma_{\bar{z}}}^2}\right)\; f_x(\bar{x},\bar{y}) \qquad (23)$$

$$\bar{E}_{\bar{y}} = \frac{\Phi_0}{\sigma_{\bar{z}}\sqrt{2\,\pi}}\exp\left(-\frac{\bar{z}^2}{2\,{\sigma_{\bar{z}}}^2}\right)\; f_y(\bar{x},\bar{y}), \qquad (24)$$

with the functions

$$f_x(\bar{x},\bar{y}) = \frac{\bar{x}}{\sigma_0}\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\,\exp\left(-(1-\xi^2)\left(\frac{\bar{x}^2}{{\sigma_0}^2}+\frac{\bar{y}^2}{{\sigma_0}^2}\frac{1}{\xi^2}\right)\right) \qquad (25)$$

$$f_y(\bar{x},\bar{y}) = \frac{\bar{y}}{\sigma_0}\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\,\frac{1}{\xi^2}\exp\left(-(1-\xi^2)\left(\frac{\bar{x}^2}{{\sigma_0}^2}+\frac{\bar{y}^2}{{\sigma_0}^2}\frac{1}{\xi^2}\right)\right). \qquad (26)$$

The functions f_x and f_y are related to the complex error function in the following way (see Appendix C):

$$f_x(\bar{x},\bar{y}) - i\,f_y(\bar{x},\bar{y}) = -i\,\left(w(\zeta_2)\,-\exp({\zeta_1}^2-{\zeta_2}^2)\,w(\zeta_1)\right) \qquad (27)$$

with

$$\zeta_1 = \frac{x}{\sigma_0}\,\kappa + i\,\frac{y}{\sigma_0}\,\frac{1}{\kappa} \quad \text{and} \quad \zeta_2 = \frac{x}{\sigma_0} + i\frac{y}{\sigma_0}. \qquad (28)$$

This establishes the connection between the fields of a 3D Gaussian bunch with the Bassetti-Erskine formula.

The linear part of the function f_x and f_y are:

$$f_{x,Lin}(\bar{x},\bar{y}) = \frac{\bar{x}}{\sigma_0}\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi = \frac{\bar{x}}{\sigma_0}\frac{2}{\sqrt{\pi}}\,(1-\kappa) \tag{29}$$

$$= \frac{\bar{x}}{\sigma_{\bar{x}}}\sqrt{\frac{2}{\pi}}\,\frac{\sigma_{\bar{x}}-\sigma_{\bar{y}}}{\sqrt{{\sigma_{\bar{x}}}^2-{\sigma_{\bar{y}}}^2}} \tag{30}$$

$$f_{y,Lin}(\bar{x},\bar{y}) = \frac{\bar{y}}{\sigma_0}\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\,\frac{1}{\xi^2} = \frac{\bar{y}}{\sigma_0}\frac{2}{\sqrt{\pi}}\,(\frac{1}{\kappa}-1) \tag{31}$$

$$= \frac{\bar{y}}{\sigma_{\bar{y}}}\sqrt{\frac{2}{\pi}}\,\frac{\sigma_{\bar{x}}-\sigma_{\bar{y}}}{\sqrt{{\sigma_{\bar{x}}}^2-{\sigma_{\bar{y}}}^2}}. \tag{32}$$

As an example the functions f_x and f_y are plotted in Fig. 2 for a Gaussian bunch with $\sigma_x = 1$ mm and $\sigma_y = 0.1$ mm. Furthermore the linear approximations of the functions are also plotted.

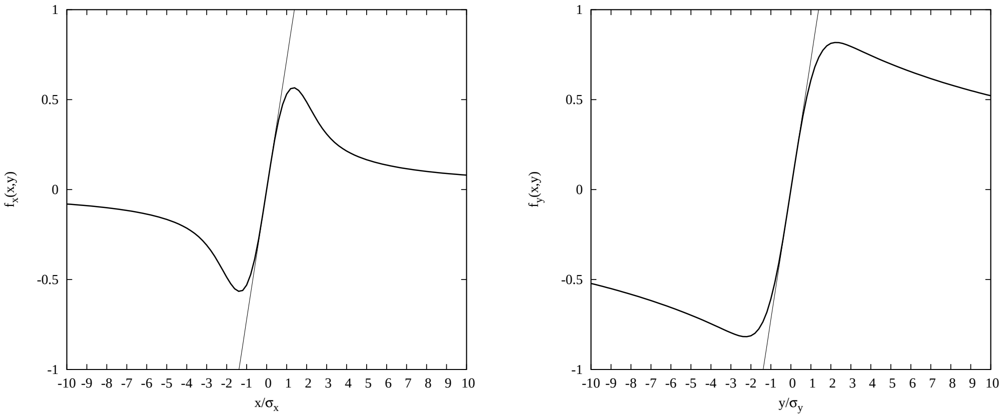

Figure 2: The functions $f_x(x,0)$, $f_y(0,y)$ and their linear approximations are plotted for a Gaussian beam with $\sigma_x = 1$ mm and $\sigma_y = 0.1$ mm versus the horizontal/vertical position. The left graph shows $f_x(x,0)$ versus the horizontal position in units of σ_x while the right graph shows $f_y(0,y)$ versus the vertical positron in units of σ_y.

2.4 Approximation of the Longitudinal Electric field

Using the same approximation as before ($\sigma_{\bar{z}} \gg \sigma_{\bar{x}} > \sigma_{\bar{y}}$) one obtains from Eqn. 18:

$$\bar{z}\,\frac{q_{\bar{x}}}{q_{\bar{z}}}\,\frac{1}{\sqrt{q_{\bar{z}}}} \sim \frac{{\sigma_0}^2}{2\,(1-\xi^2)}\,\frac{\bar{z}}{{\sigma_{\bar{z}}}^3} \sim \frac{1}{{\gamma_b}^2}. \tag{33}$$

The longitudinal electric field scales as $1/{\gamma_b}^2$ and may be neglected for many applications ($\bar{E}_{\bar{z}} \approx 0$).

Using the same approximation as in the previous section (see Eqn. (22)) one obtains:

$$\bar{E}_{\bar{z}} = \frac{\Phi_0}{\sigma_{\bar{z}}\sqrt{2\,\pi}} \exp\left(-\frac{\bar{z}^2}{2\,{\sigma_{\bar{z}}}^2}\right)\; f_{\bar{z}}(\bar{x},\bar{y},\bar{z}), \tag{34}$$

with

$$\begin{aligned} f_{\bar{z}}(\bar{x},\bar{y},\bar{z}) &= \frac{\bar{z}}{\sigma_0}\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\; h_{\bar{z}}(\xi)\; \exp\left(-(1-\xi^2)\left(\frac{\bar{x}^2}{{\sigma_0}^2}+\frac{\bar{y}^2}{{\sigma_0}^2}\frac{1}{\xi^2}\right)\right) \\ h_{\bar{z}}(\xi) &= \frac{{\sigma_0}^2}{2\,({\sigma_{\bar{z}}}^2-{\sigma_{\bar{x}}}^2)\,(1-\xi^2)+{\sigma_0}^2}. \end{aligned} \tag{35}$$

In the special case $\bar{x}=0$ and $\bar{y}=0$ the following analytic expression[1] is a good approximation of the integral:

$$\int_\kappa^1 d\xi\; h_{\bar{z}}(\xi) \approx \frac{{\sigma_0}^2}{2\,{\sigma_{\bar{z}}}^2}\ln\left(\frac{2\sqrt{2}\,\sigma_{\bar{z}}}{\sigma_0}\right), \tag{36}$$

and therefore

$$f_{\bar{z}}(0,0,\bar{z}) \approx \frac{\bar{z}}{\sigma_{\bar{z}}}\frac{2}{\sqrt{\pi}}\frac{\sigma_0}{2\,\sigma_{\bar{z}}}\ln\left(\frac{2\sqrt{2}\,\sigma_{\bar{z}}}{\sigma_0}\right). \tag{37}$$

Instead of the transformation of the variable q according to Eqn. (10) one can also use the following transformation:

$$\chi^2 = \frac{q_{\bar{y}}}{q_{\bar{z}}}, \quad \text{or} \quad q = \frac{{\sigma_1}^2}{1-\chi^2} - 2\,{\sigma_{\bar{z}}}^2, \tag{38}$$

with $\sigma_1 = \sqrt{2\,({\sigma_{\bar{z}}}^2 - {\sigma_{\bar{y}}}^2)}$. With this transformation one obtains for the longitudinal electric field an alternate representation:

$$\begin{aligned} \bar{E}_{\bar{z}} &= \Phi_0\frac{\sigma_0}{\sigma_1}\frac{\bar{z}}{\sigma_1}\frac{2}{\pi}\int_{\sigma_{\bar{y}}/\sigma_{\bar{z}}}^1 d\chi\,\frac{1}{\sqrt{q_{\bar{x}}}}\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}}-\frac{\bar{y}^2}{q_{\bar{y}}}-\frac{\bar{z}^2}{q_{\bar{z}}}\right) \\ &= \Phi_0\frac{\sigma_0\,\bar{z}}{{\sigma_1}^3}\frac{2}{\pi}\int_{\sigma_{\bar{y}}/\sigma_{\bar{z}}}^1 d\chi\sqrt{\frac{{\sigma_1}^2\,(1-\chi^2)}{{\sigma_1}^2-2\,({\sigma_{\bar{z}}}^2-{\sigma_{\bar{x}}}^2)\,(1-\chi^2)}}\exp\left(-\tfrac{\bar{x}^2}{q_{\bar{x}}}-\tfrac{\bar{y}^2}{q_{\bar{y}}}-\tfrac{\bar{z}^2}{q_{\bar{z}}}\right), \end{aligned} \tag{39}$$

[1]The result is based on the integral [4]

$$\int_0^1 d\xi\,\frac{b^2}{a^2\,(1-\xi^2)+b^2} = \frac{b^2}{a\sqrt{a^2+b^2}}\,\mathrm{arc\,tanh}\Big(\frac{a}{\sqrt{a^2+b^2}}\Big)$$

.

with

$$\begin{aligned}
&\exp\left(-\frac{\bar{x}^2}{q_{\bar{x}}}-\frac{\bar{y}^2}{q_{\bar{y}}}-\frac{\bar{z}^2}{q_{\bar{z}}}\right) \\
&= \exp\left(-(1-\chi^2)\left(\frac{\bar{x}^2}{{\sigma_1}^2-2\left({\sigma_{\bar{z}}}^2-{\sigma_{\bar{x}}}^2\right)(1-\chi^2)}+\frac{\bar{y}^2}{{\sigma_1}^2}\frac{1}{\chi^2}+\frac{\bar{z}^2}{{\sigma_1}^2}\right)\right). \quad (40)
\end{aligned}$$

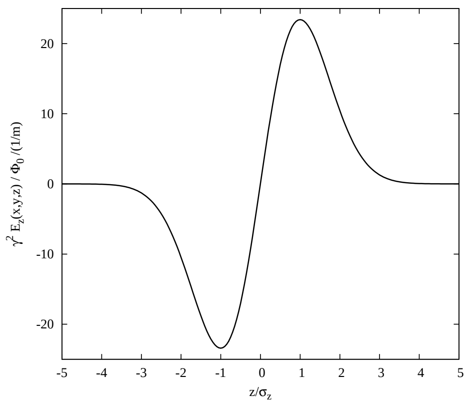

Figure 3: The longitudinal field $\bar{E}_{\bar{z}}(0,0,\bar{z})/\Phi_0$ multiplied by ${\gamma_b}^2 \approx 9.6\,10^7$ plotted for a Gaussian beam with $\sigma_x = 1$ mm, $\sigma_y = 0.1$ mm and $\sigma_{\bar{z}} = \gamma_b\,\sigma_z = 98$ m versus the longitudinal position.

3 Electric and Magnetic fields in the Laboratory frame

3.1 Lorentz Transformation of the electric field

The electric and magnetic fields in the laboratory frame can be obtained from the electric field in the rest frame of the bunch via a Lorentz Transformation, see Eqn. (6) and Appendix A.
The magnetic field is related to the electric field in the following way:

$$\begin{aligned}
B_x(x,y,z,t) &= -\frac{\beta_b}{c}\,E_y(x,y,z,t) \\
B_y(x,y,z,t) &= \frac{\beta_b}{c}\,E_x(x,y,z,t) \qquad (41) \\
B_z(x,y,z,t) &= 0.
\end{aligned}$$

For the electric field one obtains:

$$E_x(x,y,z,t) = E_0\,\frac{x}{\sigma_0}\,\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\;\; h_0(\xi)\,\exp\left(\,h_e(\xi,x,y,z,t)\,\right)$$

$$
\begin{aligned}
E_y(x,y,z,t) &= E_0\,\frac{y}{\sigma_0}\,\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\,\frac{1}{\xi^2}\,h_0(\xi)\,\exp\left(\,h_e(\xi,x,y,z,t)\,\right) \\
E_z(x,y,z,t) &= E_0\,\frac{z-\beta_b\,c\,t}{\sigma_0}\,\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\,h_z(\xi)\,h_0(\xi)\,\exp\left(\,h_e(\xi,x,y,z,t)\,\right)
\end{aligned}
\tag{42}
$$

with

$$
\begin{aligned}
\sigma_0 &= \sqrt{2\left({\sigma_x}^2-{\sigma_y}^2\right)} \\
\kappa &= \frac{\sigma_y}{\sigma_x} \\
E_0 &= \frac{1}{\sigma_z\,\sqrt{2\,\pi}}\,\frac{Q_b}{2\,\epsilon_0\,\sqrt{\pi}\,\sigma_0} \\
h_0(\xi) &= \sqrt{\frac{2\,{\gamma_b}^2\,{\sigma_z}^2\,(1-\xi^2)}{2\left({\gamma_b}^2\,{\sigma_z}^2-{\sigma_x}^2\right)(1-\xi^2)+{\sigma_0}^2}} \\
h_z(\xi) &= \frac{{\sigma_0}^2}{2\left({\gamma_b}^2\,{\sigma_z}^2-{\sigma_x}^2\right)(1-\xi^2)+{\sigma_0}^2} \\
h_e(\xi,x,y,z,t) &= -(1-\xi^2)\left(\left(\frac{x}{\sigma_0}\right)^2+\frac{1}{\xi^2}\left(\frac{y}{\sigma_0}\right)^2+\right. \\
&\qquad\left.\frac{{\gamma_b}^2\,(z-\beta_b\,c\,t)^2}{2\left({\gamma_b}^2\,{\sigma_z}^2-{\sigma_x}^2\right)(1-\xi^2)+{\sigma_0}^2}\right).
\end{aligned}
\tag{43}
$$

3.2 Approximation of the electric field

In the same approximation as in the previous section $\gamma_b \gg 1$ or $\sigma_{\bar{z}} \gg \sigma_{\bar{x}} > \sigma_{\bar{y}}$ one obtains:

$$
\begin{aligned}
E_x(x,y,z,t) &= g_z(z-\beta_b\,c\,t)\,\frac{Q_b}{2\,\epsilon_0\,\sqrt{2\,\pi}\,\sqrt{{\sigma_x}^2-{\sigma_y}^2}}\,f_x(x,y) \\
E_y(x,y,z,t) &= g_z(z-\beta_b\,c\,t)\,\frac{Q_b}{2\,\epsilon_0\,\sqrt{2\,\pi}\,\sqrt{{\sigma_x}^2-{\sigma_y}^2}}\,f_y(x,y) \\
E_z(x,y,z,t) &= g_z(z-\beta_b\,c\,t)\,\frac{Q_b}{2\,\epsilon_0\,\sqrt{2\,\pi}\,\sqrt{{\sigma_x}^2-{\sigma_y}^2}}\,f_z(x,y,z,t) \\
&\approx 0,
\end{aligned}
\tag{44}
$$

where $g_z(s)$ is the Gaussian longitudinal charge density

$$
g_z(s) = \frac{1}{\sigma_z\,\sqrt{2\,\pi}}\,\exp\left(-\frac{s^2}{2\,{\sigma_z}^2}\right),
\tag{45}
$$

and $f_x(x,y)$, $f_y(x,y)$ are the functions, which are defined in section 2.3. The function $f_z(x,y,z,t)$ is the function $f_{\bar{z}}$ of Eqn. (35) rewritten in terms of the variables of the laboratory frame ($\sigma_{\bar{z}} = \gamma_b\, \sigma_z$):

$$f_z(x,y,z,t) = \frac{z - c\,t}{\sigma_0}\,\frac{2}{\sqrt{\pi}}\int_\kappa^1 d\xi\;h_z(\xi)\;\exp\left(-(1-\xi^2)\left(\frac{x^2}{{\sigma_0}^2} + \frac{y^2}{{\sigma_0}^2}\frac{1}{\xi^2}\right)\right) \tag{46}$$

(note that $h_z(\xi) \sim 1/{\gamma_b}^2$).

The **time average of the electric fields** are:

$$\int_{-\infty}^{\infty} dt\, E_{x,y}(x,y,z,t) \;=\; \frac{1}{\beta_b\, c}\,\frac{Q_b}{2\,\epsilon_0\,\sqrt{2\,\pi}\,\sqrt{{\sigma_x}^2 - {\sigma_y}^2}}\, f_{x,y}(x,y) \tag{47}$$

$$\int_{-\infty}^{\infty} dt\, E_z(x,y,z,t) \;=\; 0. \tag{48}$$

If the linear part of the functions $f_x(x,y)$ and $f_y(x,y)$ is used (see Eqn. (30) and (32)) one obtains:

$$\int_{-\infty}^{\infty} dt\, E_x(x,y,z,t) \;=\; \frac{1}{\beta_b\, c}\,\frac{Q_b}{2\,\pi\,\epsilon_0}\;\frac{x}{\sigma_x\;(\sigma_x + \sigma_y)} \tag{49}$$

$$\int_{-\infty}^{\infty} dt\, E_y(x,y,z,t) \;=\; \frac{1}{\beta_b\, c}\,\frac{Q_b}{2\,\pi\,\epsilon_0}\;\frac{y}{\sigma_y\;(\sigma_x + \sigma_y)}. \tag{50}$$

4 Relativistic equation of motion

4.1 General considerations

To study the motion of an electron or ion in the electric field of the positron or electron bunch one has to solve the (in general) relativistic equation of motion:

$$\frac{d}{dt}\,\boldsymbol{p} = \frac{d}{dt}\,(m_0\,\gamma\,\boldsymbol{v}\,) = \boldsymbol{F}\,, \tag{51}$$

where $\boldsymbol{v}$ is the velocity, m_0 the rest mass, and

$$\gamma = \left(1 - \boldsymbol{v}\cdot\boldsymbol{v}/c^2\right)^{-1/2} \tag{52}$$

is the relativistic γ-factor of the point charge Q_0 (electron or ion) and $\boldsymbol{F}$ is the Lorentz-Force (see Eqn. 7). The time derivative of the momentum $\boldsymbol{p}$ is:

$$\frac{d}{dt}\,\boldsymbol{p} \;=\; m_0\,\gamma\,\frac{d}{dt}\boldsymbol{v} + m_0\,\gamma^3\,\boldsymbol{v}\,\frac{1}{c^2}\,(\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{v}) \tag{53}$$

$$\;=\; m_0\,\gamma\,\frac{d}{dt}\boldsymbol{v} + \boldsymbol{v}\,\frac{1}{c^2}\,(\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{p}), \tag{54}$$

where in the second equation it has been used that

$$\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{p} = m_0\,\gamma\,\left(1 + \gamma^2\,\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}\right)\,(\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{v}) = m_0\,\gamma^3\,(\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{v}). \tag{55}$$

Note that $\boldsymbol{v} \cdot \frac{d}{dt}\boldsymbol{p}$ is also equal to the time derivative of the energy $E = m_0\,\gamma\,c^2$ of the point charge:

$$\frac{d}{dt}E = \frac{d}{dt}\left(m_0\,c^2\,\gamma\right) = m_0\,\gamma^3\,(\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{v}) = \boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{p} \tag{56}$$

Now one can solve the Eqn. 54 for $d\boldsymbol{v}/dt$ and substitute the force $\boldsymbol{F}$ for $d\boldsymbol{p}/dt$:

$$\frac{d}{dt}\boldsymbol{v} = \frac{1}{m_0\,\gamma}\left(\boldsymbol{F} - \frac{1}{c^2}\,(\boldsymbol{v}\cdot\boldsymbol{F})\,\boldsymbol{v}\right) \tag{57}$$

$$= \frac{1}{m_0}\sqrt{1-\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}}\;\left(\boldsymbol{F} - \frac{1}{c^2}\,(\boldsymbol{v}\cdot\boldsymbol{F})\,\boldsymbol{v}\right) \tag{58}$$

For the Lorentz force $\boldsymbol{F} = Q_0\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$ one obtains

$$\frac{d}{dt}\boldsymbol{v} = \frac{Q_0}{m_0}\,\frac{1}{\gamma}\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{1}{c^2}\,(\boldsymbol{E}\cdot\boldsymbol{v})\,\boldsymbol{v}\right). \tag{59}$$

The vector product $\boldsymbol{v}\times\boldsymbol{B}$ can be also written as

$$\boldsymbol{v}\times\boldsymbol{B} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}\boldsymbol{v} \tag{60}$$

4.2 Motion in the space charge field of a Gaussian bunch

Now it is assumed that the point charge Q_0 is moving in the space charge field of a Gaussian bunch. The magnetic field in the laboratory is related to the electric field as (see section 3)

$$\boldsymbol{B} = \beta_b\,\frac{1}{c}\begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix}, \tag{61}$$

where $\beta_b\,c$ is the velocity of the beam along the z-axis. The equation of motion for the trajectory $\boldsymbol{r}$ of the point charge Q_0 is

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{G}(\boldsymbol{r},\boldsymbol{v},t) \end{pmatrix}, \tag{62}$$

with

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{v},t)$$
$$= \frac{Q_0}{m_0}\sqrt{1-\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}}\left(\boldsymbol{E} + \frac{\beta_b}{c}\begin{pmatrix} 0 & 0 & -E_x \\ 0 & 0 & -E_y \\ E_x & E_y & 0 \end{pmatrix}\boldsymbol{v} - \frac{1}{c^2}\,(\boldsymbol{E}\cdot\boldsymbol{v})\,\boldsymbol{v}\right) \tag{63}$$

Often it is convenient to measure the position of the point charge Q_0 in units of the dimensions of the Gaussian bunch and to use the variable $\tau = c\,t/\sigma_z$ instead of the time t. Let D be the diagonal matrix with the beam dimension on the diagonal and D^{-1} the inverse matrix of D

$$D = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \qquad D^{-1} = \begin{pmatrix} {\sigma_x}^{-1} & 0 & 0 \\ 0 & {\sigma_y}^{-1} & 0 \\ 0 & 0 & {\sigma_z}^{-1} \end{pmatrix}. \tag{64}$$

Then the normalized position is

$$\boldsymbol{\varrho} = D^{-1}\, \boldsymbol{r}\,. \tag{65}$$

With this definition one obtains:

$$\boldsymbol{r} = D\, \boldsymbol{\varrho}, \qquad \boldsymbol{v} = \frac{c}{\sigma_z}\, D\, \frac{d}{d\tau}\, \boldsymbol{\varrho}, \qquad \frac{d}{dt}\, \boldsymbol{v} = \left(\frac{c}{\sigma_z}\right)^2 D\, \frac{d^2}{d\tau^2}\, \boldsymbol{\varrho}. \tag{66}$$

The equation of motion can now be rewritten using the variables τ and $\boldsymbol{\varrho}$:

$$\frac{d}{d\tau} \begin{pmatrix} \boldsymbol{\varrho} \\ \\ \frac{d}{d\tau}\boldsymbol{\varrho} \end{pmatrix} = \begin{pmatrix} \frac{d}{d\tau}\boldsymbol{\varrho} \\ \\ \boldsymbol{G}(D\, \boldsymbol{\varrho},\; \frac{c}{\sigma_z}\, D\, \frac{d}{d\tau}\, \boldsymbol{\varrho},\; \frac{\sigma_z}{c}\, \tau) \end{pmatrix}, \tag{67}$$

with

$$\begin{aligned} &\boldsymbol{G}(D\, \boldsymbol{\varrho},\; \frac{c}{\sigma_z}\, D\, \frac{d}{d\tau}\, \boldsymbol{\varrho},\; \frac{\sigma_z}{c}\, \tau) \\ &= \frac{C_0}{\gamma} \left(\sigma_z D^{-1} \frac{\boldsymbol{E}}{E_0} + \beta_b \begin{pmatrix} 0 & 0 & -\frac{\sigma_z}{\sigma_x}\frac{E_x}{E_0} \\ 0 & 0 & -\frac{\sigma_z}{\sigma_y}\frac{E_y}{E_0} \\ \frac{\sigma_x}{\sigma_z}\frac{E_x}{E_0} & \frac{\sigma_y}{\sigma_z}\frac{E_y}{E_0} & 0 \end{pmatrix} \frac{d}{d\tau}\, \boldsymbol{\varrho} \right. \\ &\qquad \left. - \frac{1}{c^2} \left(\frac{\boldsymbol{E}}{E_0} \cdot \frac{1}{\sigma_z} D \frac{d}{d\tau}\, \boldsymbol{\varrho}\right) \frac{d}{d\tau}\, \boldsymbol{\varrho} \right), \end{aligned} \tag{68}$$

$$\frac{1}{\gamma} = \sqrt{1 - \left(\frac{\sigma_x}{\sigma_z} \frac{d}{d\tau} \varrho_x\right)^2 - \left(\frac{\sigma_y}{\sigma_z} \frac{d}{d\tau} \varrho_y\right)^2 - \left(\frac{d}{d\tau} \varrho_z\right)^2}, \tag{69}$$

and a constant C_0

$$C_0 = \frac{Q_0}{m_0\, c^2}\, \sigma_z\, E_0. \tag{70}$$

E_0 is the amplitude of the electric field, which is used to normalize the electric field strength. If the approximations from section 3 of the electric fields are used one obtains

$$E_0 = \frac{1}{\sigma_z\, \sqrt{2\,\pi}}\; \frac{Q_b}{2\, \epsilon_0\, \sqrt{\pi}\, \sigma_0}, \tag{71}$$

and

$$
\begin{aligned}
\frac{E_x}{E_0} &= \exp\left(-\frac{(\varrho_z - \beta_b\,\tau)^2}{2}\right)\, f_x(\sigma_x\,\varrho_x, \sigma_y\,\varrho_y) \\
\frac{E_y}{E_0} &= \exp\left(-\frac{(\varrho_z - \beta_b\,\tau)^2}{2}\right)\, f_y(\sigma_x\,\varrho_x, \sigma_y\,\varrho_y).
\end{aligned}
\tag{72}
$$

5 Applications

5.1 Motion of an electron in the space charge field of the bunch

In positron storage rings electrons produced by photoemission, ionization and secondary emission accumulate in the vacuum chamber forming an "electron cloud". The previously obtained formulae for the motion of a point charge in the space charge field of a Gaussian bunch are now applied to the motion of one electron of an electron cloud. It is assumed that the electron is initially at rest in the laboratory frame. Usually, the transverse kick on the electron is calculated from the ratio of the change of the transverse momentum and the longitudinal momentum: $\Delta\boldsymbol{p}_\perp/p_{||}$. Since the electron from the electron cloud may have a very small longitudinal momentum ($p_{||} \approx 0$) the following approach is adopted for the kick on the electron:

$$
\Delta\boldsymbol{r'}_\perp = \frac{1}{c}\Delta\boldsymbol{v}_\perp \approx \frac{1}{m_0\,c}\,\Delta\boldsymbol{p}_\perp = \frac{-e}{m_0\,c}\int_{-\infty}^{\infty} dt\,\boldsymbol{E}_\perp(x,y,z,t). \tag{73}
$$

Using the linear approximation for the electric field (Eqn. (50)) one obtains for the vertical kick:

$$
\Delta y' = -\frac{N_b\,e^2}{m_0\,c^2}\,\frac{1}{2\,\pi\,\epsilon_0}\,\frac{y}{\sigma_y\,(\sigma_x+\sigma_y)} = -\frac{2\,N_b\,r_e}{\sigma_x+\sigma_y}\,\frac{y}{\sigma_y}, \tag{74}
$$

for a total bunch charge of $Q_b = N_b\,e$ positrons. r_e is the classical electron radius

$$
r_e = \frac{e^2}{4\,\pi\,\epsilon_0\,m_0\,c^2} = 2.818\,10^{-15}\,\mathrm{m}. \tag{75}
$$

It has been assumed that the bunch is ultra relativistic, i.e. $\beta_b = 1$. The vertical angular oscillation frequency ω_e of the electron is

$$
\frac{\omega_{e,y}}{c} = \sqrt{\frac{N_b}{2\,\sigma_z}\,\frac{2\,r_e}{\sigma_y\,(\sigma_x+\sigma_y)}}, \tag{76}
$$

assuming an effective bunch length of $2\,\sigma_z$. The bunch acts as a thin lens with focal length $1/(2\,\sigma_z\,(\omega_{e,y}/c)^2)$:

$$\begin{pmatrix} y_f \\ y'_f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\,\sigma_z \left(\frac{\omega_{e,y}}{c}\right)^2 & 1 \end{pmatrix} \begin{pmatrix} y_i \\ y'_i \end{pmatrix}, \tag{77}$$

where (y_i, y'_i), and (y_f, y'_f) are the initial and final phase space coordinates of the electron before and after the interaction with the bunch. In the horizontal plane one obtains in analogy to the vertical plane:

$$\Delta x' = -\frac{2\,N_b\;r_e}{\sigma_x + \sigma_y}\,\frac{x}{\sigma_x}, \quad \frac{\omega_{e,x}}{c} = \sqrt{\frac{N_b}{2\,\sigma_z}\,\frac{2\,r_e}{\sigma_x\;(\sigma_x + \sigma_y)}}. \tag{78}$$

The beam parameters of KEKB-LER [2] [2] are used to calculate the kicks and the oscillation frequency of an electron in the linear approximation. The results are summarized in table 1. The kick of the positron bunch on the electron from

Parameter	Symbol	Value	Unit
Positron energy	E_b	3.5	GeV
RMS beam size	σ_x	420	μm
	σ_y	60	μm
RMS bunch length	σ_z	4	mm
Charge of one bunch	Q_b	5.3	nC
Number of positrons per bunch	N_b	$3.3 \cdot 10^{10}$	
oscillation frequency of	$\omega_{e,x}/c$	$2\pi \cdot 16.2$	GHz
an electron	$\omega_{e,y}/c$	$2\pi \cdot 42.9$	GHz
kick for $1 \times \sigma_{x,y}$ offset	$2\,N_b\,r_e/(\sigma_x + \sigma_y)$	0.387	rad

Table 1: Main parameters of KEK-LER [2] and the kicks and the oscillation frequency of an electron in the linear approximation.

the cloud is rather strong and the period of one (vertical) oscillation $2\pi\,c/\omega_{e,y}$, as calculated from the linear approximation, is only a factor 1.7 larger than the RMS bunch length σ_z. Therefore it is expected that the (nonlinear) motion of the electron will significantly differ from the result obtained from the linear kick approximation.

The (nonlinear) equation of motion according to Eqn. (67) is numerically solved with a Runge-Kutta [5] solver using a commercial code [6]. In total 1500 integration steps are used to solve the equation of motion in the interval from 0 to 15 for the variable $\tau = c\,t/\sigma_z$. The positron bunch is at the origin of the laboratory frame at time $t = 0$, while the electron is at rest at $z_0 = 5\,\sigma_z$ (again time $t = 0$). The bunch is moving with the velocity $v_b = c$ toward the electron. The situation is shown in Fig. 4. The numerical solutions of the vertical motion

[2] low energy ring of the B-factory at KEK, Tsukuba, Japan

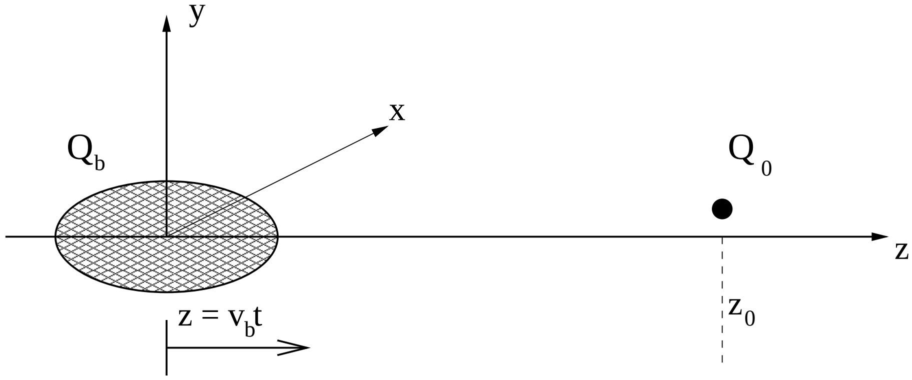

Figure 4: Initial conditions at time $t = 0$. A Gaussian bunch is moving with the velocity $v_b = c$ toward a point charge Q_0 at rest.

are shown in Fig. 5 and 6 and for the horizontal motion in Fig. 7 and 8. The equation of motion is solved in the normalized coordinates $\boldsymbol{\varrho} = D^{-1}\,\boldsymbol{r}$. The vertical kick on the electron is related to the derivative of $\boldsymbol{\varrho}$ with respect to τ in the following way:

$$\frac{d}{d\tau}\varrho_y = \frac{\sigma_z}{c}\,\frac{d}{dt}\varrho_y = \frac{\sigma_z}{c}\,\frac{1}{\sigma_y}\,\frac{d}{dt}y = \frac{\sigma_z}{c}\,\frac{1}{\sigma_y}\,c\,y'. \tag{79}$$

Therefore one obtains for the vertical and in the same way for the horizontal kick:

$$y' = \frac{\sigma_y}{\sigma_z}\,\frac{d}{d\tau}\varrho_y, \;\; x' = \frac{\sigma_x}{\sigma_z}\,\frac{d}{d\tau}\varrho_x. \tag{80}$$

The linear model for the interaction of the electron with the bunch is a very poor approximation of the electron motion since the oscillation period of the electron in the bunch potential is of the same order as the rms bunch length. There is almost no longitudinal motion of the electron. The change of the longitudinal position $(z - z_0)/\sigma_z$ is shown versus $\tau = c\,t/\sigma_z$ in Fig. 9.

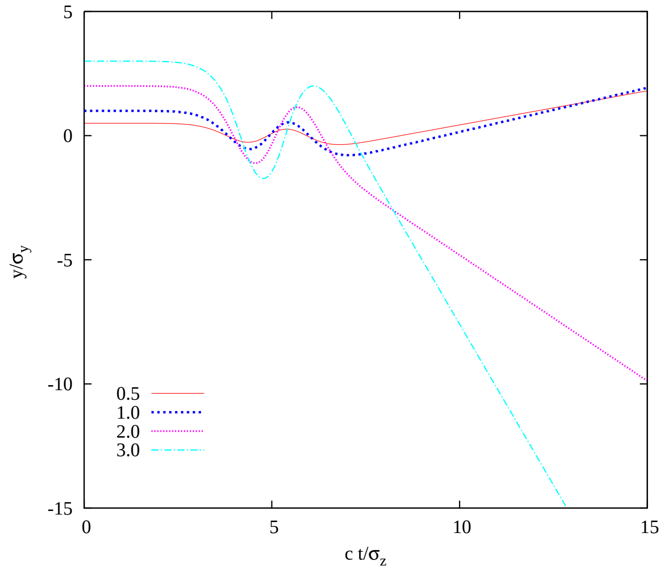

Figure 5: Vertical position of the electron versus the normalized time τ for different initial offsets of the electron (0.5, 1.0, 2.0, and $0.3 \times \sigma_y$). The initial horizontal offset is zero in all cases.

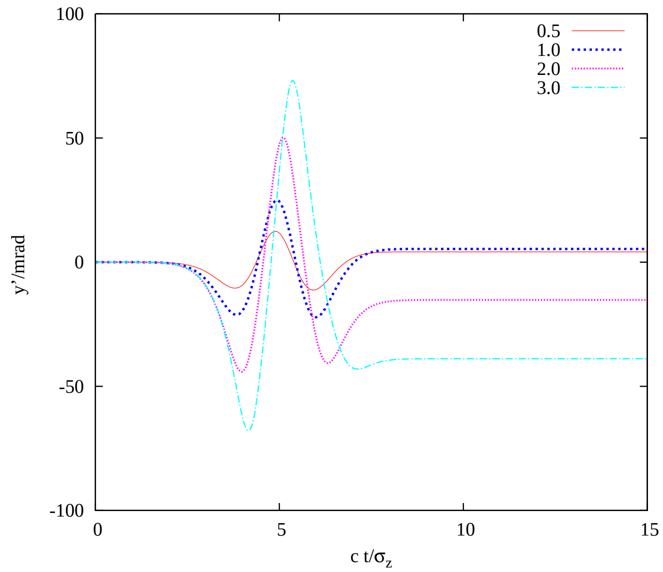

Figure 6: Vertical kick of the electron versus the normalized time τ for different initial offsets of the electron (0.5, 1.0, 2.0, and $0.3\ \times \sigma_y$). The initial horizontal offset is zero in all cases.

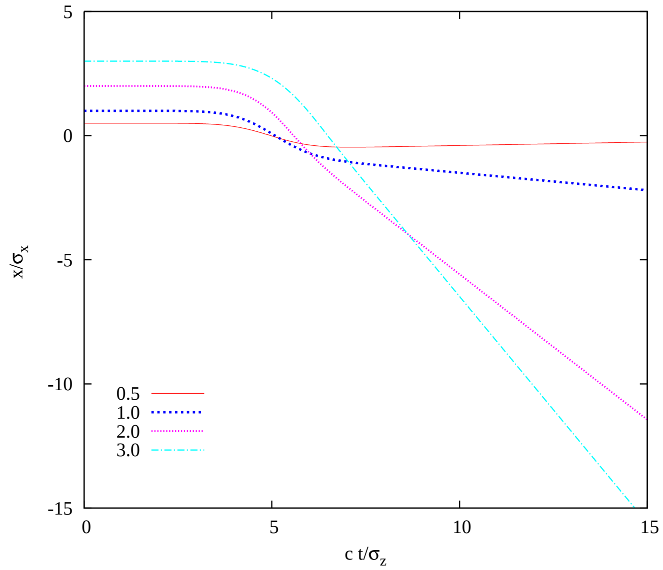

Figure 7: Horizontal position of the electron versus the normalized time τ for different initial offsets of the electron (0.5, 1.0, 2.0, and 3.0 $\times\ \sigma_x$). The initial vertical offset is zero in all cases.

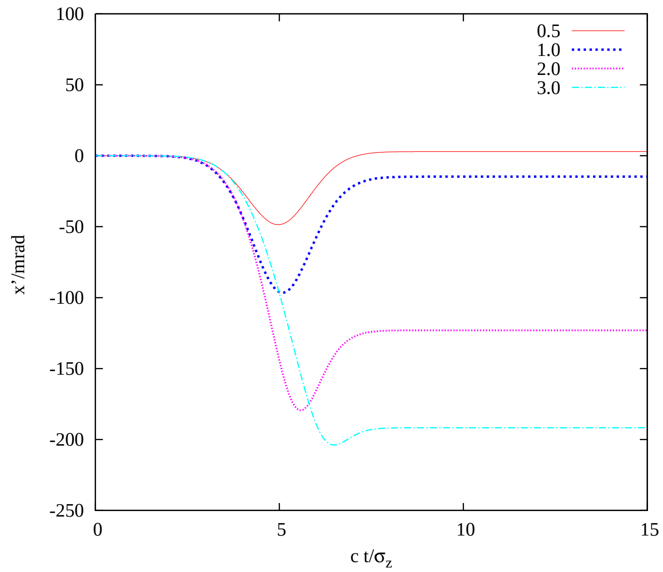

Figure 8: Horizontal kick of the electron versus the normalized time τ for different initial offsets of the electron (0.5, 1.0, 2.0, and 3.0 $\times \sigma_x$). The initial vertical offset is zero in all cases.

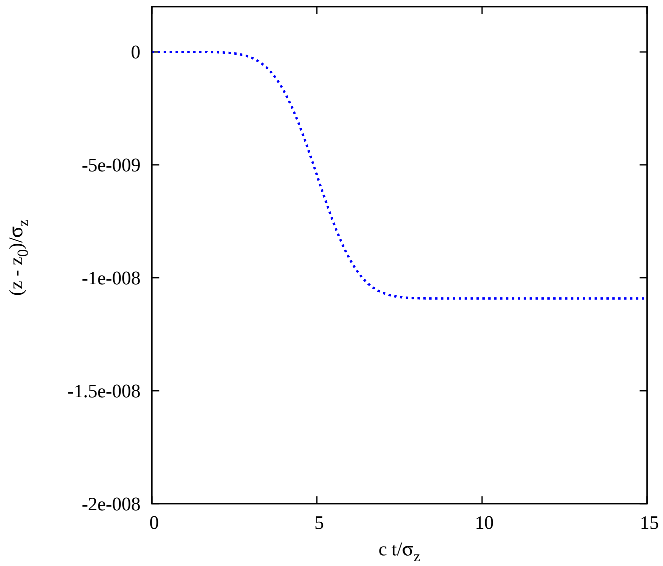

Figure 9: Longitudinal motion of the electron versus the normalized time τ for an initial offset of $1.0 \times \sigma_y$. The initial horizontal offset is zero and the initial longitudinal position z_0 is $5.0 \times \sigma_z$.

5.2 Motion of an ion in the space charge field of an electron bunch

In the vacuum chamber of an accelerator the electron bunches can ionize the residual gas molecules. Subsequently the ions may interact with the electron bunches, which may degenerate the performance of accelerator when ions are trapped in the beam potential. The kick on the ion due to the electron bunch is:

$$\Delta \boldsymbol{r'}_\perp = \frac{e}{A_{ion}\, m_p\, c} \int_{-\infty}^{\infty} dt\, \boldsymbol{E}_\perp(x,y,z,t) \tag{81}$$

$$= -\frac{2\, N_b\, r_p}{A_{ion}} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{{\sigma_x}^2 + {\sigma_y}^2}}\, \boldsymbol{f}_\perp(x,y), \tag{82}$$

where N_b is the electron bunch population, r_p is the classical proton radius

$$r_p = \frac{e^2}{4\,\pi\,\epsilon_0\, m_p\, c^2} = 1.538\, 10^{-18}\,\mathrm{m}, \tag{83}$$

m_p the proton mass and A_{ion} the atomic mass number of the ion. The function $\boldsymbol{f}_\perp(x,y) = (f_x(x,y), f_y(x,y))^t$ is the Bassetti-Erskine presentation of the transverse electric field, see section 2.3.

Using again the linear approximation for the electric field one obtains for the vertical and horizontal kick on a positively charged ion:

$$\Delta y' = -\frac{2\,N_b\,r_p}{(\sigma_x+\sigma_y)\,A_{ion}}\,\frac{y}{\sigma_y}, \quad \Delta x' = -\frac{2\,N_b\,r_p}{(\sigma_x+\sigma_y)\,A_{ion}}\,\frac{x}{\sigma_x}, \tag{84}$$

for a total bunch charge of $Q_b = -N_b\,e$ electrons. The angular oscillation frequency ω_{ion} of the ion is

$$\frac{\omega_{ion,y}}{c} = \sqrt{\frac{N_b}{2\,\sigma_z}\,\frac{2\,r_p}{A_{ion}\,\sigma_y\,(\sigma_x+\sigma_y)}}, \quad \frac{\omega_{ion,x}}{c} = \sqrt{\frac{N_b}{2\,\sigma_z}\,\frac{2\,r_p}{A_{ion}\,\sigma_x\,(\sigma_x+\sigma_y)}}. \tag{85}$$

The motion of a carbon oxide (CO) ion ($A_{ion} = 28$) is considered for the beam parameters of the synchrotron light source PETRA III, which started beam operation with positrons in 2009. But in the future PETRA III may be likely operated with electrons. The horizontal emittance is $\epsilon_x = 1\,\mathrm{nm}$ and for the simulations it is assumed that the vertical emittance is about 1 % of the horizontal emittance. The single bunch intensity will be $0.5{\cdot}10^{10}$ electrons in the 960 bunch mode and a total current of 100 mA. The important parameters for the interaction of an ion with a single bunch are summarized in table 2. The beam size has been calculated for the design emittance and the optic functions in a FODO cell. The

Parameter	Symbol	Value	Unit
Energy	E_b	6.0	GeV
RMS beam size	σ_x	1000	μm
	σ_y	10	μm
RMS bunch length	σ_z	12	mm
Charge of one bunch	Q_b	0.8	nC
Number of positrons per bunch	N_b	$0.5{\cdot}10^{10}$	
oscillation frequency of	$\omega_{ion,x}/c$	$2\pi\cdot 7.1$	MHz
an CO-ion	$\omega_{ion,y}/c$	$2\pi\cdot 71.7$	MHz
kick for $1\times\sigma_{x,y}$ offset	$2\,N_b\,r_p/(A_{ion}(\sigma_x+\sigma_y))$	0.543	μrad

Table 2: Main parameters of PETRA III [7] and the kicks and the oscillation frequency of a CO-ion in the linear approximation.

numerical solutions of the vertical motion are shown in Fig. 10 and 11 for an initial offset of the CO-ion of one σ_y. Results for the normalized horizontal and vertical position versus the time variable $\tau = c\,t/\sigma_z$ are shown in Fig. 12 and 13. Again 1500 integration steps are used to solve the equation of motion in the interval from $\tau = 0$ to $\tau = 15$ with a Runge-Kutta solver. The situation shown in Fig. 13 may be compared with the linear approximation. The electron bunch acts as a thin lens on the ion and subsequently the ion drifts over a distance of $10\,\sigma_z$. The corresponding transport matrix is:

$$\begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 1 & 10\,\sigma_z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k_0\,\frac{1}{\sigma_y} & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{86}$$

where

$$k_0 = \frac{2\,N_b\;r_p}{A_{ion}\,(\sigma_x+\sigma_y)} = 0.543\,\mu\text{rad}. \tag{87}$$

For ${y'}_0 = 0$ one obtains for the final position y_1 normalized to the initial offset y_0:

$$\frac{y_1}{y_0} = 1 - k_0\,\frac{10\,\sigma_z}{\sigma_y} = 0.993, \tag{88}$$

and under the same assumptions in the horizontal plane:

$$\frac{x_1}{x_0} = 1 - k_0\,\frac{10\,\sigma_z}{\sigma_x} = 0.999935, \tag{89}$$

The results in Fig. 12 and 13 clearly show the non linear deviations from the linear result from Eqn. (89) and (88). The relative deviation of the linear approximation from the nonlinenar Bassetti-Erskine function, i.e. the quantities (see section 2.3)

$$\frac{f_{x,Lin}(x,0)-f_x(x,0)}{f_x(x,0)} \quad \text{and} \quad \frac{f_{y,Lin}(0,y)-f_y(0,y)}{f_y(0,y)}, \tag{90}$$

are plotted in Fig. 14.

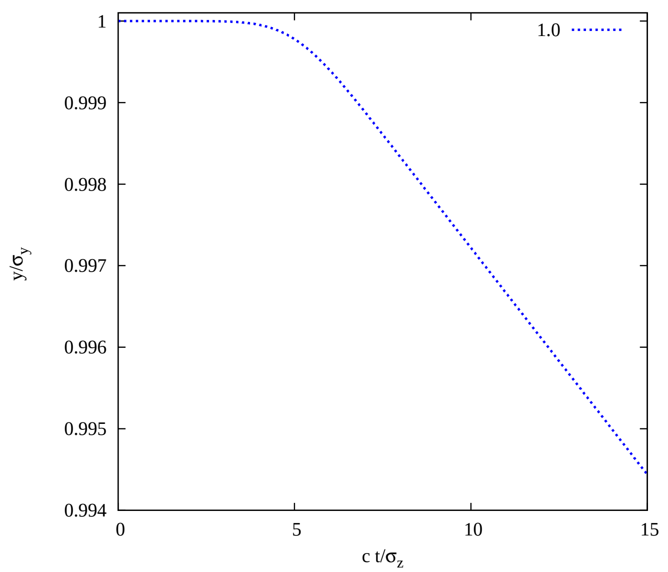

Figure 10: Vertical position of an ion versus the normalized time τ for an initial offset of one σ_y.

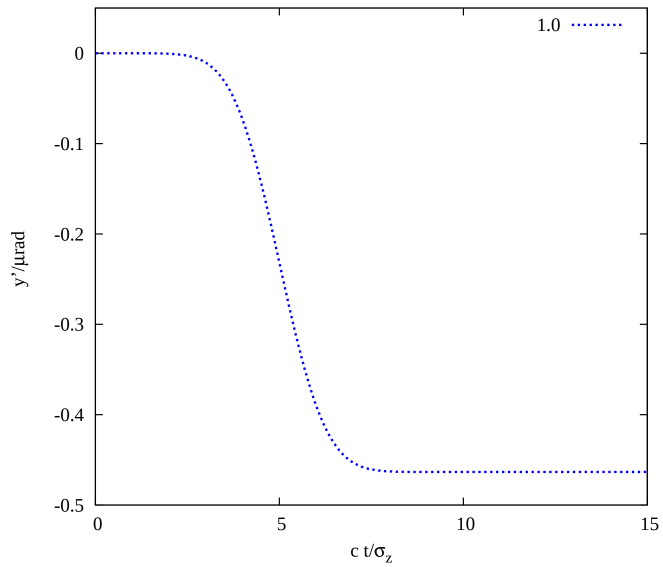

Figure 11: Vertical kick of an ion versus the normalized time τ for an initial offset of one σ_y.

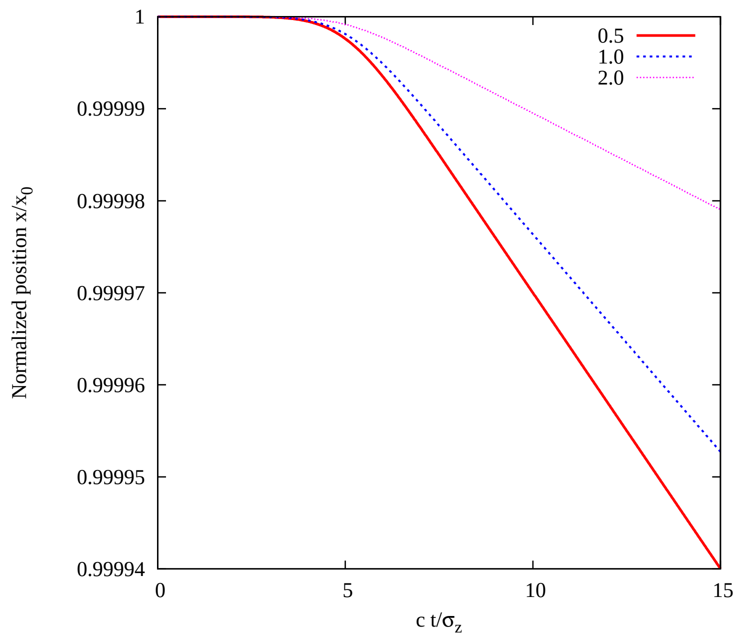

Figure 12: Normalized horizontal position of an ion versus the normalized time τ for different initial offsets x_0 (0.5, 1.0, and 2.0 $\times\ \sigma_x$).

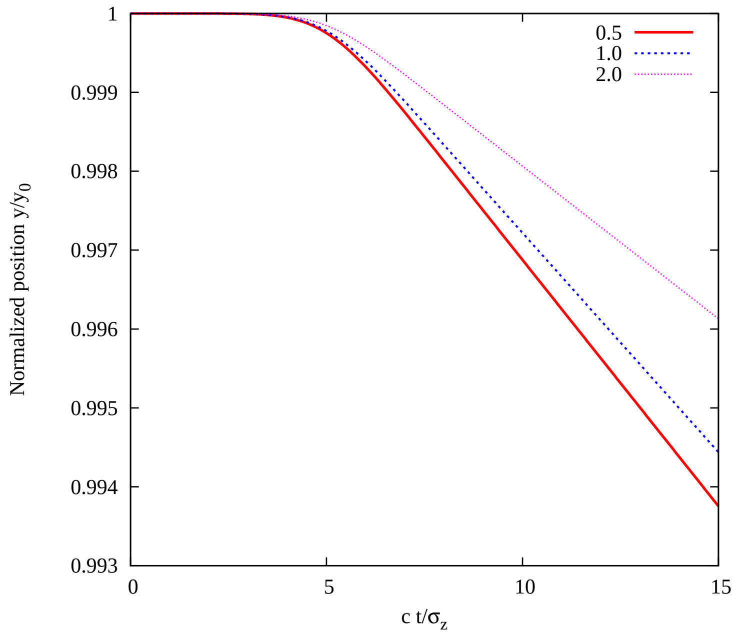

Figure 13: Normalized vertical position of an ion versus the normalized time τ for different initial offsets y_0 (0.5, 1.0, and $2.0 \times \sigma_x$).

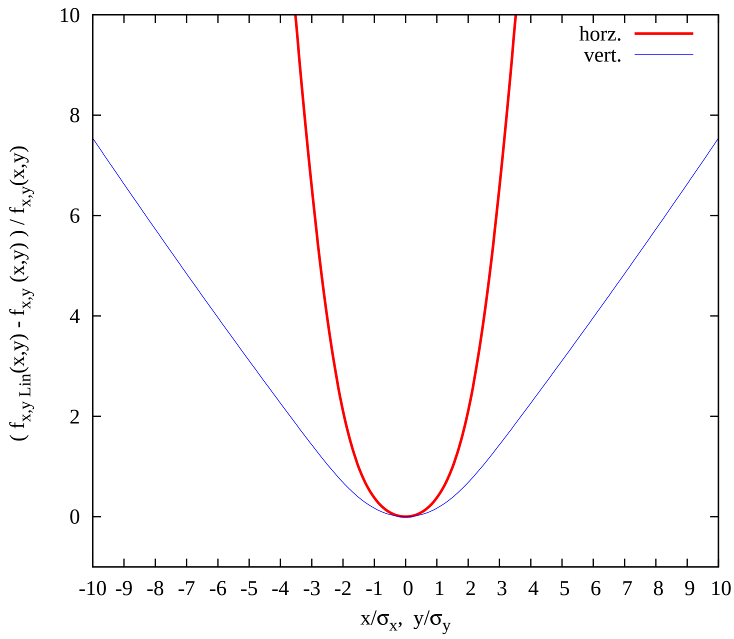

Figure 14: Deviation of the linear approximation from the nonlinear Bassetti-Erskine formula versus thr normalized horizontal and vertical position (x/σ_x and y/σ_y).

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A Lorentz Transformation

A frame $\bar{\mathrm{K}}$ is moving with the velocity v_r along the z-axis of the laboratory frame K. The coordinates and the electric and magnetic fields in both frames are related to each other via a Lorentz transformation [8].

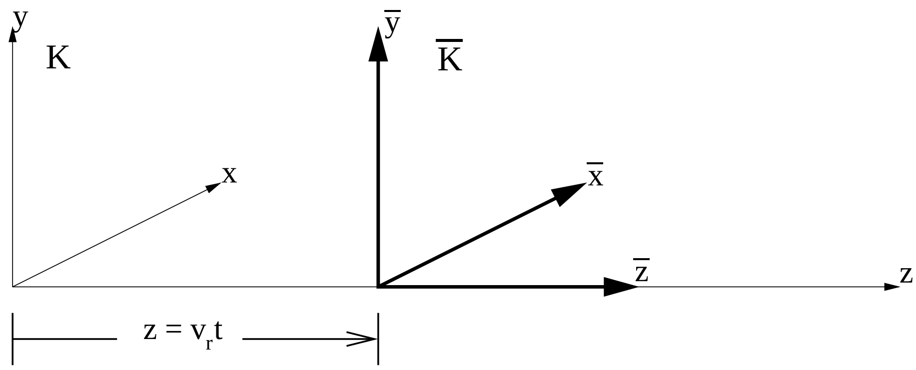

Figure 15: The reference frame $\bar{K}$ is moving with respect to the frame K in the positive z-direction with the velocity v_r.

The velocity of frame $\bar{\mathrm{K}}$ relative to frame K, and the relativistic γ-factor:

$$\boldsymbol{v_r} = \begin{pmatrix} 0 \\ 0 \\ v_r \end{pmatrix}, \qquad \beta_r = \frac{v_r}{c}, \qquad \gamma_r = \frac{1}{\sqrt{1-\beta_r^2}} \tag{91}$$

Coordinate transformation:

$$\begin{array}{rcl} c\,\bar{t} &=& \gamma_r\,(c\,t - \beta_r\,z) \\ \bar{x} &=& x \\ \bar{y} &=& y \\ \bar{z} &=& \gamma_r\,(z - \beta_r\,c\,t) \end{array} \qquad \begin{array}{rcl} c\,t &=& \gamma_r\,(c\,\bar{t} + \beta_r\,\bar{z}) \\ x &=& \bar{x} \\ y &=& \bar{y} \\ z &=& \gamma_r\,(\bar{z} + \beta_r\,c\,\bar{t}) \end{array} \tag{92}$$

transformation of electric and magnetic fields:

$$\begin{array}{rcl} \bar{E}_z &=& E_z \\ \bar{\boldsymbol{E}}_\perp &=& \gamma_r\,(\boldsymbol{E} + \boldsymbol{v_r} \times \boldsymbol{B})_\perp \\ && \\ \bar{B}_z &=& B_z \\ \bar{\boldsymbol{B}}_\perp &=& \gamma_r\left(\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v_r} \times \boldsymbol{E}\right)_\perp \end{array} \qquad \begin{array}{rcl} E_z &=& \bar{E}_z \\ \boldsymbol{E}_\perp &=& \gamma_r\left(\bar{\boldsymbol{E}} - \boldsymbol{v_r} \times \bar{\boldsymbol{B}}\right)_\perp \\ && \\ B_z &=& \bar{B}_z \\ \boldsymbol{B}_\perp &=& \gamma_r\left(\bar{\boldsymbol{B}} + \frac{1}{c^2}\boldsymbol{v_r} \times \bar{\boldsymbol{E}}\right)_\perp \end{array} \tag{93}$$

Lorentz force on charge q and velocity $\boldsymbol{v}$ in the frame K:

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{94}$$

A.1 Components

The electric field $\bar{\boldsymbol{E}}$ and magnetic field $\bar{\boldsymbol{B}}$ in the frame $\bar{\mathrm{K}}$ are transformed to the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in the laboratory frame K:

$$\boldsymbol{E} = \begin{pmatrix} \gamma_r(\bar{E}_x + v_r\,\bar{B}_y) \\ \gamma_r(\bar{E}_y - v_r\,\bar{B}_x) \\ \bar{E}_z \end{pmatrix}, \tag{95}$$

$$\boldsymbol{B} = \begin{pmatrix} \gamma_r(\bar{B}_x - v_r\,\bar{E}_y/c^2) \\ \gamma_r(\bar{B}_y + v_r\,\bar{E}_x/c^2) \\ \bar{B}_z \end{pmatrix}. \tag{96}$$

In the special case of a pure electric field in the reference frame $\bar{\mathrm{K}}$ ($\bar{\boldsymbol{B}} = 0$) one obtains:

$$\boldsymbol{E} = \begin{pmatrix} \gamma_r\,\bar{E}_x \\ \gamma_r\,\bar{E}_y \\ \bar{E}_z \end{pmatrix}, \tag{97}$$

and

$$\boldsymbol{B} = \begin{pmatrix} -\gamma_r\,v_r\,\bar{E}_y/c^2 \\ \gamma_r\,v_r\,\bar{E}_x/c^2 \\ 0 \end{pmatrix}. \tag{98}$$

The Lorentz-Force on a charge q with a velocity $\boldsymbol{u}$ in the reference frame K is:

$$\boldsymbol{F} = q \begin{pmatrix} \gamma_r\,\bar{E}_x(1 - u_z\,v_r/c^2) \\ \gamma_r\,\bar{E}_y(1 - u_z\,v_r/c^2) \\ \bar{E}_z + \gamma_r\,v_r\,(\bar{E}_x\,u_x + \bar{E}_y\,u_y)/c^2 \end{pmatrix}, \tag{99}$$

B Solution of the Poisson Equation

B.1 Integral representation of the Green's function

The solution of the Poisson equation

$$\nabla^2 \Phi(\boldsymbol{r}) = -\frac{1}{\epsilon_0} \rho(\boldsymbol{r}) \tag{100}$$

for an arbitrary charge distribution $\rho(\boldsymbol{r})$ can be written as [10, 9]

$$\Phi(\boldsymbol{r}) = \frac{1}{\epsilon_0} \int d^3\acute{r}\, G(\boldsymbol{r}, \acute{\boldsymbol{r}})\ \rho(\acute{\boldsymbol{r}}), \tag{101}$$

where

$$G(\boldsymbol{r}, \acute{\boldsymbol{r}}) = \frac{1}{4\,\pi} \frac{1}{|\boldsymbol{r} - \acute{\boldsymbol{r}}|} \tag{102}$$

is the Greens function of the Poisson equation

$$\nabla^2 G(\boldsymbol{r}, \acute{\boldsymbol{r}}) = -\delta^3(\boldsymbol{r} - \acute{\boldsymbol{r}}). \tag{103}$$

Often it is useful to rewrite the Green's function as an integral representation [11, 12, 13]:

$$G(\boldsymbol{r}, \acute{\boldsymbol{r}}) = \frac{1}{2\,\pi^{3/2}} \int_0^\infty d\xi\ \exp(-|\boldsymbol{r} - \acute{\boldsymbol{r}}|^2\, \xi^2), \tag{104}$$

using the well known integral formula $\int_0^\infty d\xi\ \exp(-x^2\,\xi^2) = \sqrt{\pi}/(2\,x)$. With the substitution $\xi = 1/\sqrt{q}$ one obtains for the Green's function [12]:

$$G(\boldsymbol{r}, \acute{\boldsymbol{r}}) = \frac{1}{4\,\pi^{3/2}} \int_0^\infty dq\, \frac{1}{q^{3/2}}\ \exp(-|\boldsymbol{r} - \acute{\boldsymbol{r}}|^2/q), \tag{105}$$

and finally for the potential $\Phi(\boldsymbol{r})$ of the charge distribution $\rho(\boldsymbol{r})$:

$$\Phi(\boldsymbol{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_0^\infty dq\, \frac{1}{q^{3/2}\,\sqrt{\pi}} \int d^3\acute{r}\, \rho(\acute{\boldsymbol{r}})\ \exp(-|\boldsymbol{r} - \acute{\boldsymbol{r}}|^2/q). \tag{106}$$

B.2 Potential of a Gaussian charge distribution

The general result of Eqn. (106) is now applied to a Gaussian charge distribution:

$$\rho(x, y, z) = \frac{Q}{(2\,\pi)^{3/2}\, \sigma_x\, \sigma_y\, \sigma_z}\ \exp\left(-\frac{x^2}{2\,{\sigma_x}^2} - \frac{y^2}{2\,{\sigma_y}^2} - \frac{z^2}{2\,{\sigma_z}^2}\right). \tag{107}$$

One obtains:

$$\int_{-\infty}^\infty d\acute{x}\, \frac{1}{\sqrt{2\,\pi}\,\sigma_x}\ \exp\left(-\frac{\acute{x}^2}{2\,{\sigma_x}^2}\right) \frac{1}{\sqrt{q}}\ \exp\left(-\frac{|x - \acute{x}|^2}{q}\right) = \frac{\exp\left(\frac{x^2}{q + 2\,{\sigma_x}^2}\right)}{\sqrt{q + 2\,{\sigma_x}^2}} \tag{108}$$

for the integration with respect to the variable $\acute{x}$. The potential of a Gaussian charge distribution is therefore:

$$\Phi(x,y,z) = \frac{Q}{4\,\pi\,\epsilon_0}\,\frac{1}{\sqrt{\pi}}\int_0^\infty dq\,\frac{\exp\left(-\frac{x^2}{q_x} - \frac{y^2}{q_y} - \frac{z^2}{q_z}\right)}{\sqrt{q_x\,q_y\,q_z}} \tag{109}$$

with

$$q_x = q + 2\,{\sigma_x}^2, \quad q_y = q + 2\,{\sigma_y}^2, \quad q_z = q + 2\,{\sigma_z}^2 \tag{110}$$

C The Bassetti Erskine Formula and the Complex Error Function

C.1 A two dimensional potential

In appendix B the potential of a three dimensional Gaussian charge distribution was obtained. A two dimensional form of the potential, which is defined as:

$$\phi(x,y) = \frac{Q'}{4\,\pi\epsilon_0}\int_0^\infty dq\,\frac{\exp\left(-\frac{x^2}{q+2\,\sigma_x{}^2}-\frac{y^2}{q+2\,\sigma_y{}^2}\right)-1}{\sqrt{q+2\,\sigma_x^2}\,\sqrt{q+2\,\sigma_y^2}}. \tag{111}$$

was used in ref. [14, 15] to study space charge forces and beam-beam effects. Q' is the longitudinal line charge density.

The corresponding electric fields are:

$$\begin{aligned} E_x &= -\frac{Q'}{4\,\pi\epsilon_0}\int_0^\infty dq\,\frac{\partial}{\partial x}\psi(x,y,q) = \frac{Q'}{4\,\pi\epsilon_0}\int_0^\infty dq\,\frac{2\,x\,\psi(x,y,q)}{\sqrt{q+2\,\sigma_x^2}} \\ E_y &= -\frac{Q'}{4\,\pi\epsilon_0}\int_0^\infty dq\,\frac{\partial}{\partial y}\psi(x,y,q) = \frac{Q'}{4\,\pi\epsilon_0}\int_0^\infty dq\,\frac{2\,y\,\psi(x,y,q)}{\sqrt{q+2\,\sigma_y^2}}, \end{aligned} \tag{112}$$

where $\psi(q,x,y)$ [3] is a two dimensional version of the integrand of Eqn.(109):

$$\psi(x,y,q) = \frac{\exp\left(-\frac{x^2}{q+2\,\sigma_x{}^2}-\frac{y^2}{q+2\,\sigma_y{}^2}\right)}{\sqrt{q+2\,\sigma_x{}^2}\,\sqrt{q+2\,\sigma_y{}^2}}. \tag{113}$$

The integrals for the electric fields can be rewritten using a **new variable** ξ [1]

$$\xi = \sqrt{\frac{q+2\,\sigma_{\bar{y}}{}^2}{q+2\,\sigma_{\bar{x}}{}^2}} \quad \text{or} \quad q = -\frac{{\sigma_0}^2}{\xi^2-1} - 2{\sigma_x}^2, \tag{114}$$

with $\sigma_0 = \sqrt{2\,({\sigma_x}^2 - {\sigma_y}^2)}$, assuming that $\sigma_x > \sigma_y$. With this transformation one obtains for the potential $\phi(x,y)$:

$$\phi(x,y) = \frac{-Q'}{2\,\pi\epsilon_0}\int_{\sigma_y/\sigma_y}^1 d\xi\,\frac{\exp\left((\xi^2-1)\frac{x^2}{{\sigma_0}^2}+(1-1/\xi^2)\frac{y^2}{{\sigma_0}^2}\right)-1}{\xi^2-1}, \tag{115}$$

[3] Note that the integral $\int_0^\infty dq\,\psi(x,y,q)$ is divergent since $\int_0^{q_{max}} dq\,1/\left(\sqrt{q+2\,\sigma_x{}^2}\,\sqrt{q+2\,\sigma_y{}^2}\right) \sim \ln(q_{max})$ is divergent for $q_{max}\to\infty$.

and for the electric fields:

$$
\begin{aligned}
E_x &= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\int_{\sigma_y/\sigma_x}^{1} d\xi\ \frac{x}{\sigma_0}\ \exp\left((\xi^2-1)\frac{x^2}{{\sigma_0}^2}\ +(1-1/\xi^2)\frac{y^2}{{\sigma_0}^2}\right)\\
&= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\ \frac{x}{\sigma_0}\int_{\sigma_y/\sigma_x}^{1} d\xi\ \exp(-(1-\xi^2))\ \exp\left(\frac{x^2}{{\sigma_0}^2}\ +\frac{1}{\xi^2}\frac{y^2}{{\sigma_0}^2}\right)\\
E_y &= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\int_{\sigma_y/\sigma_x}^{1} d\xi\ \frac{y}{\sigma_0}\ \frac{1}{\xi^2}\ \exp\left((\xi^2-1)\frac{x^2}{{\sigma_0}^2}\ +(1-1/\xi^2)\frac{y^2}{{\sigma_0}^2}\right)\\
&= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\ \frac{y}{\sigma_0}\int_{\sigma_y/\sigma_x}^{1} d\xi\ \frac{1}{\xi^2}\ \exp(-(1-\xi^2))\ \exp\left(\frac{x^2}{{\sigma_0}^2}\ +\frac{1}{\xi^2}\frac{y^2}{{\sigma_0}^2}\right).
\end{aligned}
\tag{116}
$$

C.2 The Complex Error Function

The complex error functions $\mathrm{erf}(z)$ and $w(z)$ are defined as [16] :

$$
\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z d\zeta\ \exp(-\zeta^2) \tag{117}
$$

$$
w(z) = \exp(-z^2)\left(1+\frac{2\,i}{\sqrt{\pi}}\int_0^z d\zeta\ \exp(\zeta^2)\right). \tag{118}
$$

One can rewrite $w(z)$ in terms of $\mathrm{erf}(z)$:

$$
w(z) = \exp(-z^2)\ (1-\mathrm{erf}(-i\,z)\,)\,. \tag{119}
$$

The function $w(z)$ can be used to rewrite the integral $\int d\zeta\ \exp(\zeta^2)$ in the following form:

$$
\int_{z_1}^{z_2} d\zeta\ \exp(\zeta^2) = -i\,\frac{\sqrt{\pi}}{2}\ \exp(z_2^2)\ \left(w(z_2)-\exp(z_1^2-z_2^2)\,w(z_1)\right). \tag{120}
$$

C.3 The Bassetti Erskine Formula

The electric fields from Eqn.(116) can be rewritten with the help of the complex error function $w(z)$ [1]. One obtains for the complex quantity $E_x - i\,E_y$:

$E_x(x,y) - i\,E_y(x,y)$

$$
\begin{aligned}
&= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\int_{\sigma_y/\sigma_x}^{1} d\xi\ \left(\frac{x}{\sigma_0}-i\frac{y}{\sigma_0}\,\frac{1}{\xi^2}\right)\exp\left((\xi^2-1)\frac{x^2}{{\sigma_0}^2}\ +(1-1/\xi^2)\frac{y^2}{{\sigma_0}^2}\right)\\
&= \frac{Q'}{\pi\epsilon_0\ \sigma_0}\ \exp\left(-(\frac{x}{\sigma_0}+i\,\frac{y}{\sigma_0})^2\right)\int_{\sigma_y/\sigma_x}^{1} d\xi\ \left(\frac{x}{\sigma_0}-i\frac{y}{\sigma_0}\,\frac{1}{\xi^2}\right)\exp\left((\frac{x}{\sigma_0}\xi+i\,\frac{y}{\sigma_0}\frac{1}{\xi})^2\right).
\end{aligned}
$$

The integral can be transformed into an integral in the complex plane along the path

$$\zeta = \frac{x}{\sigma_0}\xi + i\,\frac{y}{\sigma_0}\frac{1}{\xi}, \quad \text{with } \frac{\sigma_y}{\sigma_x} \le \xi \le 1 : \tag{121}$$

$$E_x(x,y) - i\,E_y(x,y) = \frac{Q'}{\pi\epsilon_0\ \sigma_0}\ \exp\left(-{z_2}^2\right)\int_{z_1}^{z_2} d\zeta\ \exp\left(\zeta^2\right), \tag{122}$$

with

$$\begin{aligned} z_1 &= \frac{x}{\sigma_0}\,\frac{\sigma_y}{\sigma_x} + i\,\frac{y}{\sigma_0}\,\frac{1}{\sigma_y/\sigma_x} \\ z_2 &= \frac{x}{\sigma_0} + i\frac{y}{\sigma_0}. \end{aligned} \tag{123}$$

Note that

$${z_1}^2 - {z_2}^2 = -\frac{x^2}{2\,{\sigma_x}^2} - \frac{y^2}{2\,{\sigma_y}^2}. \tag{124}$$

Equation (122) can be rewritten in terms of the complex error function $w(z)$ (Eqn. (120):

$$E_x(x,y) - i\,E_y(x,y) = \frac{Q'}{4\,\pi\epsilon_0}\,\frac{-i\,2\,\sqrt{\pi}}{\sigma_0}\,\left(w(z_2)\ - \exp({z_1}^2 - {z_2}^2)\,w(z_1)\right),\ \text{or} \tag{125}$$

$$E_x(x,y) = \frac{Q'}{2\,\epsilon_0\sqrt{\pi}\sigma_0}\ \mathrm{Im}\left(w(z_2)\ - \exp({z_1}^2 - {z_2}^2)\,w(z_1)\right) \tag{126}$$

$$E_y(x,y) = \frac{Q'}{2\,\epsilon_0\sqrt{\pi}\sigma_0}\ \mathrm{Re}\left(w(z_2)\ - \exp({z_1}^2 - {z_2}^2)\,w(z_1)\right). \tag{127}$$