Exercises week 5:

Problem 1:

$$\nabla_\mu V_\nu - \nabla_\nu V_\mu = \partial_\mu V_\nu - \partial_\nu V_\mu - \Gamma^\rho{}_{\mu\nu} V_\rho + \Gamma^\rho{}_{\nu\mu} V_\rho$$

$\Gamma^\lambda{}_{\mu\nu}$ is symmetric in μ and ν

$$\Rightarrow \nabla_\mu V_\nu - \nabla_\nu V_\mu = \partial_\mu V_\nu - \partial_\nu V_\mu$$

Problem 2:

$$\Gamma^\mu{}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left[\frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \underbrace{\frac{\partial g_{\lambda\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho}}_{=0} \right]$$

$$= \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^\lambda}$$

Using the relationship for symmetric matrices

$$\partial \operatorname{Tr} \log M = \operatorname{Tr} M^{-1} \partial M$$

and $\log \det M = \operatorname{tr} \log M$

$$\Rightarrow \operatorname{Tr} M^{-1} \partial M = \partial \log \det M$$

$$\Rightarrow \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^\lambda} = \frac{1}{2} \partial_\lambda \log(g) = \frac{1}{\sqrt{g}} \partial_\lambda \sqrt{g}$$

[Notice a subtlety here with

$$\log(-g) = \log g + \log(-1) = \log g + \text{const}$$

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The two matrix relationships can be proven via the series expansion of the matrices:

$$\log(1-x) = -\sum_{n>0} \frac{1}{n} x^n \qquad (|x|<1)$$

$$\log M = \log(\mathbb{1} - (\mathbb{1} - M)) = -\sum_{n>0} \frac{1}{n} (\mathbb{1} - M)^n$$

$$(M^n)' = M' M^{n-1} + M M' M^{n-2} \ldots M^{n-1} M'$$

But in the trace, the terms can be reshuffled

$$\Rightarrow \partial \operatorname{Tr} \log M = \operatorname{Tr} \sum_{n \geq 0} (\mathbb{1} - M)^n \partial M$$

geometric series:

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n$$

$$\partial \operatorname{Tr} \log M = \operatorname{Tr} M^{-1} \partial M$$

Notice that this in principle assumes that the eigenvalues of M are smaller than one, but this requirement can be removed by just rescaling M

$$\log M = \log \epsilon M / \epsilon = \log \epsilon M - \log \epsilon$$

The other relation can be shown by diagonalizing M with an orthonormal O

$$O \bar{M} O^T = M$$

$$\hookrightarrow \log \det M = \log \det O \bar{M} O^T$$

$$= \log \det \bar{M}$$

$$= \log \prod_i m_i$$

where m_i are the eigenvalues of M.

$$\Rightarrow \log \prod_i m_i = \sum_i \log m_i = \mathrm{Tr} \log \bar{M}$$

$$= \mathrm{Tr}\, O (\log \bar{M}) O^T = \mathrm{Tr} \log (O \bar{M} O^T)$$

$$= \mathrm{Tr} \log M$$

The last relation follows from the series expansion in matrices and the relation

$$O \bar{M}^n O^T = (O \bar{M} O^T)^n$$

Problem 3:

$$\epsilon_{\mu\nu\lambda\kappa}/\sqrt{g} = \epsilon^{\alpha\beta\gamma\delta} g_{\alpha\mu} g_{\beta\nu} g_{\gamma\lambda} g_{\delta\kappa}/\sqrt{g}$$

The right hand side is totally antisymmetric and has to be proportional to the Levi-Civita symbol. The prefactor is fixed by contraction (see lecture)

$$\epsilon_{\mu\nu\lambda\kappa}/\sqrt{g} = -\sqrt{g}\,\epsilon^{\mu\nu\lambda\kappa}$$

Using this relationship, we find

$$\epsilon^{\mu\nu\lambda\kappa}\epsilon_{\mu\nu\lambda\rho}/g = -\epsilon^{\mu\nu\lambda\kappa}\epsilon^{\mu\nu\lambda\rho}$$

Notice that this is not a tensor relation and should be understood elementwise.

On the right hand side, the two free indices κ, ρ have to be the same, the prefactor is the number of combinations for 3 out of 4.

$$\to \epsilon^{\mu\nu\lambda\kappa}\epsilon_{\mu\nu\lambda\rho}/g = -6\,\delta^{\kappa}_{\rho}$$

which is a tensor relation.

Problem 4:

$$\nabla_\alpha \left(\epsilon^{\mu\nu\kappa\lambda} / \sqrt{g} \right) = \epsilon^{\mu\nu\kappa\lambda} \partial_\alpha \frac{1}{\sqrt{g}}$$
$$+ \Gamma^\mu_{\alpha\rho} \epsilon^{\rho\nu\kappa\lambda} / \sqrt{g}$$
$$+ \Gamma^\nu_{\alpha\rho} \epsilon^{\mu\rho\kappa\lambda} / \sqrt{g}$$
$$+ \Gamma^\kappa_{\alpha\rho} \epsilon^{\mu\nu\rho\lambda} / \sqrt{g}$$
$$+ \Gamma^\lambda_{\alpha\rho} \epsilon^{\mu\nu\kappa\rho} / \sqrt{g}$$

The right side is totally antisymmetric again.
The proportionality is fixed by
contraction with $\epsilon_{\mu\nu\kappa\lambda} / \sqrt{g}$

$$-\sqrt{g}\, 24\, \partial_\alpha \frac{1}{\sqrt{g}} - 4\, \Gamma^\mu_{\alpha\mu} \cdot 6$$
$$= -24 \sqrt{g}\, \partial_\alpha \frac{1}{\sqrt{g}} - 24\, \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} = 0$$

This result is intuitively clear:
For every spacetime point one can go into
the free falling frame. The metric reads

$$g_{\mu\nu}(y) = g_{\mu\nu}(x) + (y-x)^\alpha \partial_\alpha g_{\mu\nu} + \frac{1}{2}(y-x)^\alpha (y-x)^\beta \partial_\alpha \partial_\beta g_{\mu\nu}$$

$= \eta_{\mu\nu}$ in free falling frame

$= 0$ in free falling frame

contained in $R^\lambda{}_{\lambda\nu\kappa}$ cannot be completely removed

In any case, in the free falling frame

$$\Gamma^\mu{}_{\alpha\beta} = 0 \quad \text{and} \quad \partial_\alpha g_{\mu\nu} = 0$$

$$\Rightarrow \quad \nabla_\alpha \left(\epsilon^{\mu\nu\lambda\kappa} / \sqrt{g} \right) = 0$$

and hence in all frames since the relation is a covariant.