Exercises week 4: problem 1:

$$\left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2} g^{\lambda\rho} \left\{ \frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right\}$$

$$\left\{ {\lambda \atop \mu\nu} \right\}' = \frac{1}{2} g^{\lambda\rho\,\prime} \left\{ \qquad \right\}'$$

$$= \frac{\partial g'}{\partial x'}$$

$$= \left[\frac{\partial}{\partial x'} \left(\frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} g \right) \right] \times 3$$

$$= \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \left\{ {\alpha \atop \beta\gamma} \right\} + \text{othe}$$

first term:

$$g^{\lambda\rho\,\prime} \left(\frac{\partial}{\partial x'^\mu} \frac{\partial x^\alpha \partial x^\beta}{\partial x'^\nu \partial x'^\rho} \right) g_{\alpha\beta}$$

$$g^{\lambda\rho\,\prime} \left(\frac{\partial x^\alpha}{\partial x'^\nu \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\rho} + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\rho \partial x'^\mu} \right) g_{\alpha\beta}$$

Symmetrisation:

$$(g^{\lambda\rho})' \left(\frac{\partial x^\alpha}{\partial x'^\nu \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\rho} \right) g_{\alpha\beta}$$

$$\delta^\gamma_\alpha \quad g^{\gamma\rho} \frac{\partial x'^\lambda}{\partial x^\gamma} \underbrace{\frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial x^\beta}{\partial x'^\rho}}_{\delta^\beta_\sigma} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\nu \partial x'^\mu} = \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu \partial x'^\nu}$$

$$g_{\alpha\sigma}$$

problem 2:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\partial x^\alpha \partial x^\beta}{\partial\tau\partial\tau}$$

use: $\left(\frac{dx^\mu}{d\tau}\right)' = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial\tau}$

$$\frac{\partial x'^\mu}{\partial x^\alpha}\left(\frac{\partial^2 x^\alpha}{(\partial\tau)^2} + \Gamma^\alpha{}_{\rho\sigma}\frac{\partial x^\rho \partial x^\sigma}{\partial\tau\partial\tau}\right)$$

$$+ \frac{\partial x'^\mu}{\partial x^\alpha \partial x^\gamma}\frac{\partial x^\gamma}{\partial\tau}\frac{\partial x^\alpha}{\partial\tau} + \frac{\partial x'^\mu \partial x^\rho}{\partial x^\rho \partial x'^\alpha \partial x'^\beta}\frac{\partial x'^\alpha}{\partial\tau}\frac{\partial x'^\beta}{\partial\tau}$$

$= 0$ due to

$$\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} = -\frac{\partial x'^\kappa}{\partial x^\alpha}\frac{\partial x'^\lambda}{\partial x^\beta}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\kappa \partial x'^\lambda}$$

problem 3:

$$(\nabla_\mu V^\kappa)' = \frac{\partial}{\partial x^{\mu\prime}}\left(\frac{\partial x'^\kappa}{\partial x^\alpha} V^\alpha\right)$$

$$+ \Gamma'^\kappa_{\mu\nu} V^{\nu\prime}$$

$$= \frac{\partial x'^\kappa}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \nabla_\alpha V^\beta$$

$$+ \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial^2 x'^\kappa}{\partial x^\alpha \partial x^\beta} V^\alpha$$

$$+ \frac{\partial x'^\kappa}{\partial x^S} \frac{\partial x^S}{\partial x'^\mu \partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\sigma} V^\sigma$$

$$= \frac{\partial x'^\kappa}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \nabla_\alpha V^\beta$$

problem 4:

$$(\nabla_\kappa V_\alpha)' = \frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\beta}{\partial x'^\alpha} V_\beta\right) - \Gamma'^\beta_{\kappa\alpha} V'_\beta$$

$$= \frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\beta}{\partial x'^\alpha}\nabla_\rho V_\beta$$

$$+ \frac{\partial x^\beta}{\partial x'^\alpha \partial x'^\kappa} V_\beta$$

$$- \frac{\partial x'^\beta}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\kappa \partial x'^\alpha} V'_\beta$$

$$= \frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\beta}{\partial x'^\alpha} D_\rho V_\beta$$

problem 5:

$$\nabla_\mu g_{\alpha\beta} = 0$$

$$0 = \partial_\mu g_{\alpha\beta} - \Gamma^{\rho}_{\mu\beta} g_{\rho\alpha} - \Gamma^{\rho}_{\mu\alpha} g_{\rho\beta}$$

$$\Gamma^{\rho}_{\mu\beta} g_{\rho\alpha} = \frac{1}{2}\left\{ \cancel{\frac{\partial g_{\alpha\mu}}{\partial x^\beta}} + \frac{\partial g_{\alpha\beta}}{\partial x^\mu} - \cancel{\frac{\partial g_{\mu\beta}}{\partial x^\alpha}} \right\}$$

$$+ \quad \alpha \leftrightarrow \beta \quad = \qquad \frac{\partial g_{\alpha\beta}}{\partial x^\mu}$$

$$= \frac{\partial g_{\alpha\beta}}{\partial x^\mu}$$

$$= \partial_\mu g_{\alpha\beta} - \partial_\mu g_{\alpha\beta} = 0$$