exercises - week 3

Problem 1:
For a general symmetric matrix, there are orthonormal transformations, such that:

$$S' = O S O^T$$

S' diagonal.

Use D with diagonal elements of absolute values of S'

$$S'' = D^{-1/2} S' D^{-1/2}$$

$$= \begin{pmatrix} 1 & & \\ & \ddots 0 & \\ & & -1 \end{pmatrix}$$

$$D^{-1/2} O S O^T D^{-1/2}$$

$$= \underbrace{(D^{-1/2} O)}_{\Lambda} S \underbrace{(D^{-1/2} O)^T}_{\Lambda^T}$$

$$g'_{\mu\nu} = \Lambda^{\beta}{}_{\nu} \Lambda^{\alpha}{}_{\mu} g_{\alpha\beta}$$

In physics: The coordinate transformations will not change the signature of the metric.

Uniqueness?

$$P g P^T = \eta \qquad \text{Minkowski}$$

$$\Lambda \eta \Lambda^T = \eta$$

Any $P' = \Lambda P$ with Λ Lorentz trafo will give

$$P' g P'^T = \eta$$

Problem 2:

$$\frac{\partial}{\partial\lambda}\left(\left(\frac{ds}{d\lambda}\right)^2\right) = 0$$

$$\frac{\partial}{\partial\lambda}\left(g_{\mu\nu} \frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}\right)$$

$$= \frac{\partial x^\rho}{\partial\lambda}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda} + 2 g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial\lambda^2}\frac{\partial x^\nu}{\partial\lambda}$$

$$\frac{\partial^2 x^\mu}{\partial\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\lambda}{\partial\lambda} = 0$$

$$= \left(\Gamma^\alpha_{\mu\rho} g_{\alpha\nu} + \Gamma^\alpha_{\nu\rho} g_{\mu\alpha}\right) - 2 g_{\mu\nu} \Gamma^\mu_{\alpha\beta}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial x^\beta}{\partial\lambda}$$

$$\times \frac{\partial x^\rho}{\partial\lambda}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}$$

$$= 0 \Rightarrow \left(\frac{\partial s}{\partial \lambda}\right)^2 = \text{const}$$

$$ds^2 \propto d\lambda^2 \Rightarrow \boxed{\tau \propto \lambda}$$

$$ds^2 = -d\tau^2 \qquad \frac{dx^\mu}{\partial \lambda} \longrightarrow \frac{dx^\mu}{d\tau} \cdot \text{const}$$

So for the geodesic,
the path parameter
is alwasy proportional
to the proper time.
It is fixed by the initial
conditions and the normalization

$$\frac{\partial x^\mu}{\partial \lambda} \frac{\partial x_\mu}{\partial \lambda} = -1$$

will lead to a geodesic parameterized by the
proper time.

problem 3:

$$S_E = \frac{1}{2} \int d\lambda \; g_{\mu\nu}(x^\mu(\lambda)) \frac{dx^\mu}{d\lambda}(\lambda) \frac{dx^\nu}{d\lambda}(\lambda)$$

$$S_E\Big|_{g(x)} : \; x^\mu(\lambda) \rightarrow \mathbb{R}$$

Euler - Lagrange:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\mu}{\partial \lambda}\right)}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{\partial x^\mu}{\partial \lambda} \frac{d x^\nu}{\partial \lambda} - \frac{d}{d\lambda}\left(2 g_{\mu\alpha} \frac{\partial x^\mu}{\partial \lambda}\right)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} - 2 \frac{\partial g_{\mu\alpha}}{\partial x^\beta} \frac{\partial x^\beta}{\partial x} \frac{\partial x^\mu}{\partial \lambda}$$

$$- 2 g_{\mu\alpha} \frac{\partial^2 x^\mu}{\partial x^2}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta} + \Gamma^\beta{}_{\mu\alpha} g_{\beta\nu}$$

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left\{ \frac{\partial g_{\beta\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\nu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right\}$$

$$\Gamma^\mu{}_{\alpha\beta} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda} = \frac{1}{2} g^{\mu\nu} \left\{ 2 \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right\} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda}$$

$$\hookrightarrow -2 \frac{d^2 x^\mu}{d\lambda^2} g_{\mu\nu} - 2 \Gamma^\mu{}_{\alpha\beta} \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda} g_{\mu\nu} = 0$$

problem 4:

$$S_L = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \int d\lambda' \sqrt{-g_{\mu\nu}\frac{dx^\mu}{\partial\lambda'}\frac{dx^\nu}{\partial\lambda'}}$$

Euler-Lagrange:

$$\frac{1}{2}\frac{-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}}{\sqrt{\quad}} - \frac{d}{d\lambda}\left[\frac{-g_{\mu\alpha}\frac{dx^\mu}{d\lambda}}{\sqrt{\quad}}\right]$$

Check how it transform under reparameterizations:

$$\frac{\partial}{\partial\lambda} \to \frac{\partial\lambda'}{\partial\lambda}\frac{\partial}{\partial\lambda'}$$

Consider again the e.o.m of problem 3:

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{\partial x^\mu}{\partial\lambda}\frac{dx^\nu}{\partial\lambda} - \frac{d}{d\lambda}\left(2g_{\mu\alpha}\frac{\partial x^\mu}{\partial\lambda}\right)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{\partial x^\mu}{\partial\lambda'}\frac{\partial x^\nu}{\partial\lambda'}\left(\frac{d\lambda'}{d\lambda}\right)^2 - \frac{\partial\lambda'}{\partial\lambda}\frac{\partial}{\partial\lambda'}\left(2g_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda'}\frac{\partial\lambda'}{\partial\lambda}\right)$$

$\hookrightarrow$ $\dots + \frac{\partial\lambda'}{\partial\lambda}2g_{\mu\nu}\frac{\partial^2\lambda}{(\partial\lambda')^2}\frac{\partial x^\mu}{\partial\lambda'}$

$\hookrightarrow$ not reparameterization invariant back to problem 4:

$$-\frac{1}{2}\frac{\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}}{\sqrt{\quad}} - \frac{d}{d\lambda}\left[\frac{-g_{\mu\alpha}\frac{dx^\mu}{d\lambda}}{\sqrt{\quad}}\right]$$

$$\|$$

$$\frac{\partial\lambda'}{\partial\lambda}\left(\qquad\qquad\right)' - \frac{\partial\lambda'}{\partial\lambda}\frac{d}{d\lambda'}\left[\frac{-g_{\mu\nu}\frac{dx^\mu}{\partial\lambda'}}{\sqrt{g_{\mu\nu}\frac{dx}{\partial\lambda'}\frac{dx}{\partial\lambda'}}}\right]$$

So this equation is reparameterization invariant just as the action.

For a solution to the geodesic equation, the λ derivative on the square root vanishes.

$\rightarrow$ a solution to the geodesic equation also solves the e.o.m of action (4).

On the other hand, any reparameterization of a solution to the geodesic equation, will solve the e.o.m from (4) but not the e.o.m from (3).