problem 1:

$$\partial_\mu F^{\mu\nu} = -j^\nu$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -j^\nu$$

with Lorenz gauge:

$$\partial_\mu A^\mu = 0 \Rightarrow \quad \Box A^\mu = -j^\mu$$

for a plane wave:

$$A_\mu = \epsilon_\mu e^{i k_\lambda x^\lambda} + c.c$$

$$\rightarrow \quad k_\mu k^\mu A_\nu = 0 \quad \rightarrow \boxed{k_\mu k^\mu = 0}$$

gauge trafo:

$$A_\mu \rightarrow A_\mu' = A_\mu + \partial_\mu \alpha(x^\mu)$$

$$\alpha(x) = -i \bar{\alpha} e^{ikx} + c.c$$

$$\rightarrow \quad \epsilon_\mu \rightarrow \epsilon_\mu + k_\mu \bar{\alpha}$$

-> only 2 out of 4 degrees of freedom survive

E.g. suppose $\vec{k} \parallel z$

$$k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix} \; ; \quad k_\mu = \begin{pmatrix} -k \\ 0 \\ 0 \\ k \end{pmatrix}$$

$$A_\mu = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix} \rightarrow k^\mu A_\mu = 0$$

$$\Rightarrow A_0 + A_z = 0$$

$$A_\mu = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ -A_0 \end{pmatrix} = \begin{pmatrix} e_0 \\ e_x \\ e_y \\ -e_0 \end{pmatrix} e^{ikx}$$

residual gauge: $e_\mu \rightarrow e_\mu + \alpha k_\mu$

you can use the residual gauge dof to remove A0

$$A_\mu = \begin{pmatrix} 0 \\ A_x \\ A_y \\ 0 \end{pmatrix}$$

Here Ax and Ay would be the two transverse degrees of freedom.

problem 2: $\vec{k} \parallel z$ $\quad k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$

$$A_\mu = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

rotation around the z-axis:

$$A_\mu \to \Lambda_\mu^{\ \nu} A_\nu \qquad \left(\Lambda_\mu^{\ \nu} = \frac{\partial x^\nu}{\partial x'^\mu} \right)$$

$$\Lambda_\mu^{\ \nu} = \begin{pmatrix} 1 & & & \\ & c & s & \\ & -s & c & \\ & & & 1 \end{pmatrix} \qquad \begin{matrix} c = \cos\alpha \\ s = \sin\alpha \end{matrix}$$

$$\left.\begin{matrix} A_0 \to A_0 \\ A_z \to A_z \end{matrix}\right\} \quad h = 0$$

$$\begin{matrix} A_x \to c A_x + s A_y \\ A_y \to c A_y - s A_x \end{matrix} \quad \Big| \; SO(2)$$

$$\left.\begin{matrix} A_+ = A_x + i A_y \\ A_- = A_x - i A_y \end{matrix}\right\} \quad \text{circular polarization}$$

$$A_+ \to c A_x + s A_y + i c A_y - s i A_x$$

$$e^{i\alpha h} A_+ \to c\,(A_x + i A_y) + i s\,(A_x + i A_y)$$

$$\bar{c} = c \quad ; \quad s = -\bar{s} \quad \Rightarrow \quad \boxed{h = -1}$$

$$c = \cos\alpha \qquad s = \sin\alpha$$
$$\bar{c} = \cos h\alpha \qquad \bar{s} = \sin h\alpha$$

$$A_- \to \boxed{h = +1}$$

The circular polarizations are the helicity eigenstates with h = +- 1

$$A_{+/-} \to A_{+/-}\, e^{i\alpha(-/+)}$$

$$SO(2) \simeq SU(1)$$

problem 3:

$$h_{\mu\nu} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}$$

$$h_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta h_{\alpha\beta}$$

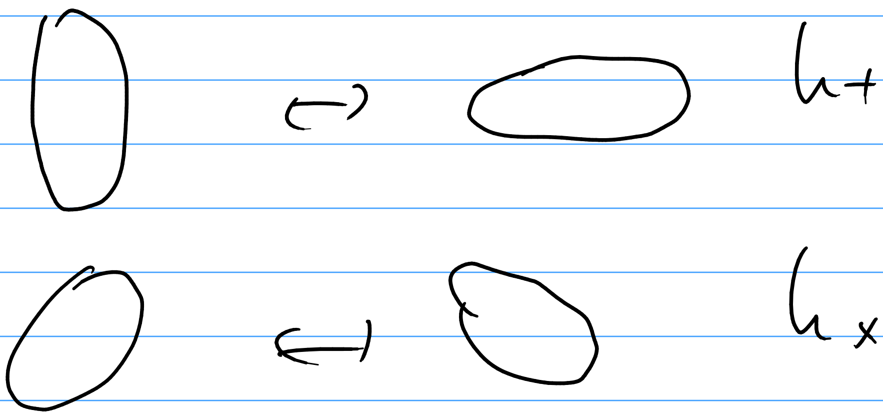
$$O \cdot h \cdot O^T$$

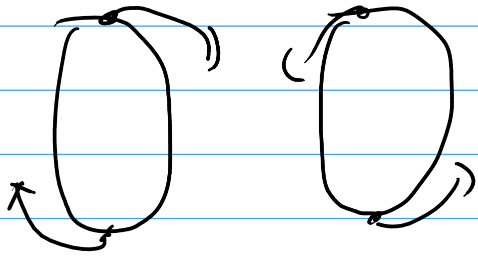
$$\Rightarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} \to \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}$$

$$(h_+ \pm i h_\times) \to e^{2i\alpha(\mp)} (h_+ \pm i h_\times)$$

These are again the circular polarizations
and their helicities are +-2.





circular polarizations