Exercises, week 9

1 Kruskal coordiantes

Starting from the isotropic metric in standard form

$$d\tau^2 = \left[1 - \frac{2GM}{r}\right] dt^2 - \left[1 - \frac{2GM}{r}\right]^{-1} dr^2 - r^2 d\Omega^2 \,, \tag{1}$$

show that using Kruskal coordinates, defined by (T is an arbitrary constant)

$$\begin{aligned} r'^2 - t'^2 &= T^2 \left(\frac{r}{2GM} - 1\right) \exp \frac{r}{2GM} \,, \\ \frac{2r't'}{r'^2 + t'^2} &= \tanh \frac{t}{2MG} \end{aligned} \tag{2}$$

the metric reads

$$d\tau^2 = \left(\frac{32G^3M^3}{rT^2}\right) \exp\left(\frac{-r}{2GM}\right) (dt'^2 - dr'^2) - r^2 d\Omega^2 \,. \tag{3}$$

The r in the last term is understood as beeing a function of r' and t'. What is the domain of the coordinates r' and t' for $r > 0$? What happened to the singularities at $r = 0$ and $r = 2GM$? Is the metric singular in these points?

2 Geodesics in a Schwarzschild background

Starting from the Schwarzschild metric

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{4}$$

determine all non-vanishing Christoffel symbols. Using the Christoffel symbols, write down the four geodesic equations.

3 Geodesics in a Schwarzschild background 2

Starting from the Schwarzschild metric (4), write down explicitly the action of the geodesic equation

$$S = \frac{1}{2} \int d\lambda \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} g_{\mu\nu} \,, \tag{5}$$

by inserting the metric.

Derive the geodesic equation using the Euler-Lagrange equations on the resulting action for $r(\lambda), t(\lambda), \theta(\lambda)$ and $\phi(\lambda)$.